SPH simulation of fluid-structure interaction problems

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Research project

- Problem: deformation of a plate due to the action of a fluid (large displacement of the structure, fluid free surface)

- Numerical technique:

  SPH  \rightarrow  \text{Lagrangian: it automatically follows moving interfaces}

  Relatively easy to include the simulation of different materials
Scheme of the model

Structure (SPH)  Fluid (SPH)

Coupling conditions on the interface:

**Kinematic condition:**
\[
\vec{v}_f = \vec{v}_s \\
\vec{v}_f \cdot \hat{n} = \vec{v}_s \cdot \hat{n} \quad \text{(perfect fluid)}
\]

**Dynamic condition:**
\[
\sigma_s \hat{n}_s = -\sigma_f \hat{n}_f \\
\hat{n}_s^T \sigma_s \hat{n}_s = p_f \quad \text{(perfect fluid)}
\]
Continuum equations

\[
\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \quad (i=1,3) \quad \text{continuity equation}
\]

\[
\rho \frac{Dv_i}{Dt} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i,j=1,3) \quad \text{equation of motion}
\]

(isothermal conditions)

\[\rho\] density
\[t\] time
\[x_i\] position (\(i\)-component of the vector)
\[v_i\] velocity (\(i\)-component of the vector)
\[\sigma_{ij}\] stress (\(ij\)-component of the tensor)
\[g_i\] gravity acceleration (\(i\)-component of the vector)
State equation and constitutive equations

**Deviatoric Stress**

\[ \sigma_{ij} = -p \delta_{ij} + S_{ij} \quad (i,j=1,3) \]

**Fluid**

\[ c_0 = \sqrt{\frac{\varepsilon}{\rho}}_0 \quad (\varepsilon \text{ compressibility modulus}) \]

**Solid**

\[ c_0 = \sqrt{\frac{k}{\rho}}_0 \quad (k \text{ bulk modulus}) \]

**Fluid**

\[ S_{ij} = 0 \quad \text{(perfect fluid)} \]

**Solid**

\[ \frac{dS_{ij}}{dt} = 2\mu \left( \dot{\varepsilon}_{ij} - \frac{1}{3} \delta_{ij} \dot{\varepsilon} \right) + S_{ik} \Omega_{jk} + \Omega_{ik} S_{kj} \]

\[ \dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]

\[ \Omega_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \quad (\mu \text{ shear modulus}) \]
2D SPH equations (1)

- **Equation of state:**
  \[ p_a = c_0^2 \left( \rho_a - \rho_{0a} \right) \]

- **Continuity equation:**
  \[ \frac{D\rho_a}{Dt} = \sum_b m_b (\vec{v}_a - \vec{v}_b) \cdot \nabla_a W_{ab} \]

(“a” and “b”: particle labels)
2D SPH equations (2)

Equation of motion:

\[
\begin{align*}
\text{fluid} & \quad \frac{Dv_{ia}}{Dt} = -\sum_b m_b \left( \frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} \right) \delta_{ij} \frac{\partial W_{ab}}{\partial x_{ja}} + g_i \\
\text{solid} & \quad \frac{Dv_{ia}}{Dt} = \sum_b m_b \left( \frac{\sigma_{ij}}{\rho_a^2} + \frac{\sigma_{ij}}{\rho_b^2} + \Pi_{ab} \delta_{ij} + R_{ijab} f^n \right) \frac{\partial W_{ab}}{\partial x_{ja}} + g_i
\end{align*}
\]

\[
\sigma_{ija} = -p_a \delta_{ij} + S_{ija}
\]

\(S_{ij}\) calculated by an implicit scheme from the incremental hypoelastic relation
2D SPH equations (3)

Velocity gradient:

\[
\frac{\partial v_i}{\partial x_j} \bigg|_a = \sum_b \frac{m_b}{\rho_b} (v_{ib} - v_{ia}) \frac{\partial W_{ab}}{\partial x_{ja}}
\]

[or: velocity gradient normalized to account for non uniformity in particle distribution]
Fluid-structure interaction

- Two sets of particles, fluid and solid

- A simple approach: to consider particles in the equations regardless of the fact they are fluid or solid

- Interpenetration of fluid and solid particles can be prevented using XSPH but the interaction is not well reproduced (excessive adhesion)

- Definition of the fluid-solid interface (and normal)
- Dynamic condition (action-reaction principle)
- Kinematic condition
Definition of the fluid-solid interface

Since solid particles maintain the same regular spatial distribution (no fragmentation):

\(\hat{t}_a = (t_{ax}, t_{ay}) = \left( \frac{x_{a+1} - x_{a-1}}{|\bar{x}_{a+1} - \bar{x}_{a-1}|}, \frac{y_{a+1} - y_{a-1}}{|\bar{x}_{a+1} - \bar{x}_{a-1}|} \right)\)

\(\hat{n}_a = (-t_{ay}, t_{ax})\)

\(\bar{x}_{int_a} = \bar{x}_a + \frac{d}{2} \hat{n}_a\)
Dynamic condition

\[ p_{\text{int}_a} = \frac{\sum_{b \in \Omega_f} \frac{m_b}{\rho_b} p_b W \left( \bar{x}_{\text{int}_a} - \bar{x}_b, h \right)}{\sum_{b \in \Omega_f} \frac{m_b}{\rho_b} W \left( \bar{x}_{\text{int}_a} - \bar{x}_b, h \right)} \]

0.5 (constant, in order to take in account possible separation of the two media)

surface term:

\[ F_{f \rightarrow s} = p_{\text{int}_a} \int_{\Gamma} W \left( \bar{x}_{\text{int}_a} - \bar{x}, h \right) d\Gamma' \]

\[ F_{s \rightarrow f,a^*} = - F_{f \rightarrow s,a} \] (linear interpolation)
Kinematic condition

- “a” fluid particle → “b*” solid particle (the nearest)

- velocity of the interface:

\[
v_{i \text{int}_{b^*}} = \frac{\sum_{b \in \Omega_s} \frac{m_b}{\rho_b} v_{ib} W\left(|\bar{x}_{\text{int}_{b^*}} - \bar{x}_b|, h\right)}{\sum_{b \in \Omega_s} \frac{m_b}{\rho_b} W\left(|\bar{x}_{\text{int}_{b^*}} - \bar{x}_b|, h\right)}
\]

- a velocity distribution is assigned to solid particles in order to obtain by SPH interpolation (on the interface) the interface velocity (normal component):

\[
v_{n_{b}} = v_{n_{\text{int}_{b^*}}} + d^* \left(v_{n_{\text{int}_{b^*}}} - v_{n_{a}}\right)
\]

\[d^* = \max\left(d_b/d_a, 2\right)\]

\[
v_{t_{b}} = v_{t_{a}}
\]
Other features of the code

• Correction of velocity:
  - XSPH (solid) (Monaghan 1989)
  - dissipative correction (fluid) (Gallati and Braschi 2000)
    (on the interface particles from both media have to be included in the correction)

• Boundary conditions:
  - fluid
    - imaginary particles which reflect velocity
    - layer of fixed particles (2h)
  - solid (clamp)
    - layer of fixed particles which are calculated just like others but with velocity equal to zero

• Time integration scheme: Euler explicit (staggered)

• Kernel: cubic spline (Monaghan 1992)
Example: Elastic gate

<table>
<thead>
<tr>
<th>Dimensions</th>
<th></th>
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<tbody>
<tr>
<td>A</td>
<td>0.1 m</td>
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<tr>
<td>H</td>
<td>0.14 m</td>
</tr>
<tr>
<td>L</td>
<td>0.079 m</td>
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<tr>
<td>S</td>
<td>0.005 m</td>
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Sluice-gate (rubber)

<table>
<thead>
<tr>
<th>( \rho_s )</th>
<th>1100 kg/m(^3)</th>
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</thead>
</table>

\( E \) (Young modulus) \( \equiv 10^7 \) Pa
Simulation

\( \varepsilon = 2 \times 10^6 \text{ N/m}^2 \) (compressible)
\( \rho_f = 1000 \text{ kg/m}^3 \)

\( K = 2 \times 10^7 \text{ N/m}^2 \)
\( \mu = 4.27 \times 10^6 \text{ N/m}^2 \)
\( \nu = 0.4 \) (Poisson coefficient)
\( E = 1.2 \times 10^7 \text{ N/m}^2 \)

\( h/d = 1.5 \)
\( n_p = 6012 \)
\( dt = 8.34 \times 10^{-6} \text{ s} \)

\( t = 0 \text{ s} \)
Comparison between simulation and experiment

$t=0 \ s$
t = 0.04 s
t=0.08 s
t=0.12 s
t=0.16 s
$t=0.2 \text{ s}$
t=0.24 s
t=0.32 s
t=0.4 s
Free end of the plate: displacements (1)
Free end of the plate: displacements (2)

vertical displacement

<table>
<thead>
<tr>
<th>time (s)</th>
<th>v.d. (m)</th>
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<tbody>
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<tr>
<td>0.4</td>
<td>0.04</td>
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</table>

- Experiment
- Simulation
Free surface (1)

water level behind the gate

- y (m)
- t (s)
- experiment
- simulation
Free surface (2)

water level 5 cm far from the gate
Future work

- To complete the normalization of the elastic equations with explicit treatment of boundary conditions
- To improve the model in order to manage larger deformations
- To write the code in cylindrical coordinates in order to simulate axially symmetric problems (“flex-flow” valves, sloshing in cylindrical tanks…)
- To use two different spatial discretizations for the fluid and the solid (membranes, hemodynamics)

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