Simulation and Modelling of a Skewed Turbulent Channel Flow

RICHARD J.A. HOWARD¹ and NEIL D. SANDHAM²
¹Équipe MOST, LEGI, BP 53, INP Grenoble, 38041 Grenoble Cedex 9, France
²Aeronautics & Astronautics, School of Engineering Sciences, University of Southampton, Southampton SO17 1BJ, U.K.

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Abstract. A time-dependent three-dimensionally skewed flow is investigated using direct numerical simulations of the incompressible Navier–Stokes equations. The effect on the instantaneous and mean turbulent field is investigated. Instantaneous flowfields reveal that the skewing has the effect of initially reducing the strength and height of quasi-streamwise vortices of both signs of rotation with respect to the skewing. A mechanism for this process is put forward. The mean flowfields show drops in turbulence quantities such as turbulence kinetic energy. In addition to this, two-equation turbulence modelling of the flow is carried out. This highlights a deficiency, in that the standard turbulence models are unable to capture the drop in turbulence intensity due to the skewing. A modification based on the exact dissipation equation is found to significantly improve the model behaviour for this flow.

Key words: skew flow, DNS, turbulence modelling.

1. Introduction

Mean three-dimensional boundary layer flows are defined as flows whose mean velocity vector varies in direction across the layer. There has been a considerable amount of research carried out in order to understand how mean three-dimensional turbulent flows behave. An objective of this research is to improve the statistical turbulence modelling of these flows. However, by taking only averaged quantities, the instantaneous structure of the flow is hidden and some of the underlying flow physics may be obscured. Therefore, an additional objective of the work is to study the relationship between flow structures and statistics of turbulence.

There have been several experiments of flows with mean three-dimensionality. Johnston and Flack [14] provide a useful review of advances in understanding three-dimensional turbulent boundary layers covering both experimental and computational studies. Some key experiments will be discussed below.

Some of the major insights into three-dimensional turbulent flow were given by Bradshaw and Pontikos [2] in their study of the turbulent flow over an infinite swept wing. In this flow they observed that the shear stress angle is not aligned with the velocity gradient angle. They also observed drops in the turbulence levels
due to the three-dimensionality. In terms of the instantaneous structure of the flow they suggested that, since the large eddies were the most efficient structures for extracting kinetic energy from the mean shear, drops in the levels of turbulence must have been caused by changes in the behaviour of the large eddies. This led them to propose a mechanism of “toppling” of the large eddies by the spanwise pressure gradient. Olçmen and Simpson [22] studied an idealised wing/body junction turbulent flow. They also observed variation in the alignment of the shear stress and strain rate vectors but recorded turbulence kinetic energy increases near the wall and decreases further from the wall. Schwarz and Bradshaw [30] looked at a three-dimensional turbulent boundary layer in a 30° bend. They found that there was a lag between the shear stress and velocity gradient vectors and also a reduction in the stress energy ratio, the ratio of the shear stress magnitude to twice the turbulent energy.

In terms of turbulence structure, Eaton [8] examined the behaviour of an embedded streamwise vortex subjected to mean flow three-dimensionality. He defined two types of quasi-streamwise vortices that could be present in a skewed flow, as shown in Figure 1. Case 1 vortices have a vortex-induced velocity near the wall in the same direction as the crossflow, while case 2 vortices have the opposite sense of rotation. Eaton concluded that crossflow should “reduce or eliminate the ability of case 1 vortices to produce ejections of low-momentum fluid away from the wall”. He also proposed that the crossflow diverted fluid as it was lifted by the case 2 vortices resulting in “decreased height reached by bubble clusters interacting with case 2 vortices”. However, Flack [9] examined turbulent flow in a 30° bend and found that the three-dimensionality did not cause ejection events associated with one sign of vortex rotation to be larger than ejection events of the opposite sign.

There have also been several direct simulations of three-dimensional mean flows. Moin et al. [21] studied the effect of applying a spanwise body force on a steady channel flow (two-dimensional mean flow) producing a transient three-dimensional mean flow. They observed reductions in turbulence kinetic energy and differences in the strain and shear stress angles. Further structural analysis of this flow was carried out by Sendstad and Moin [31]. They identified breakup of the near wall streaks and differences in the evolution of quasi-streamwise vortices with positive and negative signs (with respect to the skewing). Figure 1 includes a schematic of the proposal of Sendstad and Moin [31]. They proposed that “when the spanwise mean flow is developed, the vortices are shifted in the spanwise direction relative to the near-wall streaks. They then pump high-speed fluid down into the low-speed streak and up from the high-speed streaks. Both streaks are thus weakened”. Coleman et al. [5] studied the effect of spanwise disturbances on steady channel flow. A shear driven case arose from setting the walls of the channel into motion in the spanwise direction. They pointed out that, provided there was a uniform acceleration, this has exactly the same effect as introducing a body force in the spanwise direction. A constant body force can be considered as constant acceleration of the walls in the opposite direction. This is not the
same as the spanwise pressure gradient observed in the swept wing experiment of Bradshaw and Pontikos [2]. In order to produce a more representative effect of pressure gradient Coleman et al. [5, 6] proposed a transverse irrotational strain which “stretched” the flow. There were several important conclusions from this work. Firstly, they also observed the “torn” nature of the streaks in response to the spanwise deformation but proposed that the effect of the three-dimensionality was to modify the interaction between the quasi-streamwise vortices and streaks rather than the vortices themselves due to the fact that the main shearing region of the flow was at $x_C^+ = 10$ whereas the location of the quasi-streamwise vortices was at $x_C^+ = 20$. Secondly, all the time-developing three-dimensional mean flows showed reductions in turbulent kinetic energy and Reynolds stress. Thirdly, they found that the outer layer shear was responsible for alterations in turbulence structure. A further implication was that at higher Reynolds numbers, where the maximum crossflow velocity moves outwards in wall units, the near-wall skewed turbulence structure phenomena would be less important. Testing the models for this flow at an order of magnitude higher Reynolds number would resolve this.
issue, but this is beyond the capability of present computers. For the present study we fix the Reynolds number and focus on the effects of complex distortions to the turbulence. An important continuation of the work introduced here would be to identify Reynolds number effects.

In this paper DNS is used to provide data both to increase the understanding of a skewed turbulent flow and to improve the turbulence modelling of the flow. Explanation of the skewed turbulent flow DNS is given initially. This provides a comprehensive database which is then used to examine the effect of the skewing on turbulent flow structures. The same flow is subsequently calculated using several different two-equation turbulence models. Turbulence statistics from the DNS database are then compared to the results produced by the turbulence models. The models are found to have significant errors in predicting this flow. For this reason model modifications are investigated. A model modification is found which improves the performance of all the models for this flow and this result is highlighted in the closing section.

2. DNS of a Skewed Flow

The incompressible Navier-Stokes equations are solved using a Fourier–Chebyshev spectral code written for use on a parallel machine (in this case a Cray T3D in Edinburgh) and set up for a channel flow geometry. Time discretisation is carried out using a compact third order Runge–Kutta scheme for the convective terms and the Crank–Nicolson scheme for the viscous and pressure terms. Details of the code can be found in [28]. The Reynolds number for the plane channel flow was set to $Re = u_c h / v = 180$, where $u_c = \sqrt{\tau_w / \rho}$ is the friction velocity with $\tau_w$ the wall shear stress and $\rho$ the density. The channel half width is $h$ and the kinematic viscosity $v$. The computations use $128 \times 128 \times 80$ grid points in a box size of $13 \times 6 \times 2$ channel half widths in the streamwise, spanwise and wall normal directions respectively. It is possible that, during deformation, the flow may become less well resolved as the resolution at right angles to the mean flow will reduce as the flow skews. Two-point correlations, energy spectra and energy budgets, discussed in [10], demonstrated that this does not significantly degrade the results.

The turbulent channel flow is made three-dimensional in the mean by moving the upper and lower walls of the channel in the spanwise direction with constant velocity. The walls are moved in the same direction which allows an additional check for the simulation, as the statistical behaviour of the flow should be the same on each wall. Statistics are obtained from the simulations by spatial averaging in homogeneous directions (streamwise and spanwise) and ensemble averaging between 8 simulations each started with independent initial conditions. Figure 2 shows a schematic representation of the flow. The time has been non-dimensionalised with a reference time scale based on the channel half width and the mean friction velocity of the initial plane channel flow.
In the limit $t \to \infty$ the flow becomes a simple Galilean transform of the original two-dimensional flow. However, before reaching this stage, the flow goes through two different stages: a rapid initial response to the wall effect, followed by a very gradual recovery. The walls are moved at $u_3 = 10$ which is a rather vigorous input compared with the channel centreline velocity $u_{1c} \approx 18$. The near-wall behaviour is more rapidly affected than the flow towards the centre of the channel. The effect eventually propagates to the channel centreline and the flow reaches a transitory three-dimensional state, described as a collateral state by Coleman et al. [5]. However this flow differs from that of Coleman et al. [5] since both walls are moved instead of just one. Instead the flow is similar to that of Sendstad and Moin [31]. From this stage the flow deforms very slowly so that the streamwise velocity component returns to its original value since the spanwise velocity field contains no shear, $\partial u_3 / \partial x_2$, in the limit $t \to \infty$. The region of interest in this work is the initial, strongly three-dimensional, stage of the flow evolution.

A spanwise velocity profile is specified initially rather than having a discontinuity when the wall velocity is set. The velocity field specified is a Stokes solution for an impulsively started flat plate,

$$u_3 / u_{30} = \text{erf} \, \eta,$$

where $\eta = x_2 / 2\sqrt{t_0}$ with the wall normal direction, $x_2$, the viscosity $\nu$. For the initial condition the time is set to $t_0 = 0.0045$. This is the same input as was applied by Coleman et al. [5] in their simulation of the response of channel flow to spanwise movement of one wall. It was verified that the results were independent of the exact choice of $t_0$. The initial behaviour of the spanwise mean velocity is the same as the behaviour of the Stokes flow.

Figure 3 shows the behaviour of the spanwise mean velocity relative to the wall as compared with the Stokes solution and a rescaled, two-dimensional mean, turbulent channel flow velocity profile. From Figure 3 it can be seen that the spanwise profiles are beginning to differ from the Stokes solution at $t = 0.3$ but at $t = 2.0$, which is at or near the collateral state, the spanwise velocity profile bears no resemblance to the Stokes solution and closely matches the turbulent profile. In the limit $t \to \infty$ the spanwise velocity relative to the wall tends to
zero but, as discussed earlier, the flow reaches this state very gradually from the collateral state. The reason why the spanwise velocity profile has changed from a Stokes solution at $t = 0.3$ to a turbulent profile at $t = 2.0$ is due to the growth in the spanwise shear stress component, $\bar{u}_2\bar{u}_3$, or rather the reorganisation of the turbulence in the skewed direction and the corresponding development of viscous and semi-logarithmic regions in the velocity profile in that direction.

An indication of the lack of equilibrium can be obtained from the relative angles of the shear stress

$$\gamma_r = \arctan\left(\frac{\bar{u}_1\bar{u}_3}{\bar{u}_1\bar{u}_2}\right),$$  \hspace{1cm} (2)

and strain rate

$$\gamma_s = \arctan\left(\frac{\partial \bar{u}_3/\partial x_2}{\partial \bar{u}_1/\partial x_2}\right).$$  \hspace{1cm} (3)

In an equilibrium flow the strain rate direction is equal to the shear stress direction so there is no difference between the two angles. Figure 4 shows the distributions of shear stress and strain rate angles at $t = 0.3$ and $t = 2.0$. At $t = 0.3$ it can be seen that there is a small but non-negligible lag between the shear stress and the strain-rate angles in the near-wall region. However, at $t = 2.0$, the shear stress and strain rate angles are almost the same at about 30 degrees. This means that the flow can be described as in a kind of equilibrium with a stress and strain field rotated approximately 30 degrees to the original flow direction.
3. Statistics

Another important feature of this three-dimensional flow is the significant drop in the turbulence levels due to the disturbance. Figure 5 shows the variation of the peak (over $x_2$) in the turbulence kinetic energy $K = (u'_i u'_j)/2$ with time. This figure shows that the peak in the turbulence kinetic energy drops by approximately 35% at $t = 0.3$. Figure 5 also shows the variation of the streamwise wall shear velocity $u_r = \sqrt{v(\partial u_1/\partial x_2)_{\text{wall}}}$. This also decreases with time but at a different rate compared to the turbulence kinetic energy. This gives an indication that the drop in turbulence levels is different at different points within the channel. Figure 6 shows the turbulence kinetic energy budgets for plane channel flow and the
Figure 6. Turbulence kinetic energy budgets for plane channel flow (a) and the skewed flow at $t = 0.3$ (b) and $t = 2.0$ (c).
skew flow at $t = 0.3$ and $t = 2.0$ there $P_k$ is production, $\varepsilon$ is dissipation, $J_K^p$ is pressure transport, $J_K^v$ is viscous transport and $J_K^u$ is transport involving triple moments. These plots give indications of the extent to which the turbulence decays at $t = 0.3$ and subsequently builds up again by $t = 2.0$. It is also clear that the distribution of the budget at $t = 2.0$ is similar to that of the original plane channel flow indicating that the turbulence is returning to plane channel flow conditions.

A more comprehensive analysis can be given by looking at the behaviour of the full Reynolds stress budgets this is carried out in [10]. All the components of the Reynolds stress tensor are non-zero in this flow and, since the Reynolds stress tensor is symmetric, there are six different budgets for each time.

4. Instantaneous Visualisation

A feature of the shear driven flow is the breakup of the near wall streaks [31]. These are elongated lumps of fluid that move slower than the rest of the fluid (i.e. where $u' < 0$) and are found for $x^+ < 50$. The breakup of the streaks is initially unexpected as normally one would think that the streaks would merely turn in the direction of the mean flow in response to the three-dimensionality. As was discussed in the introduction, there was a suggestion that the three-dimensionality of the flow tended to enhance vortices of one sign and disrupt vortices of the opposite sign. The main purpose of this section is to try to improve the understanding of the mechanisms within this flow that cause things such as streak breakup and shed some light on the effect of three-dimensionality on quasi-streamwise vortices.

One of the problems involved in studies of turbulence structure is the method of identifying and locating structures such as quasi-streamwise vortices [4, 12]. No perfect measure has been found but, in the current work, local low pressure regions (i.e. where $p' < 0$) have been found to give a useful indication of where vortices are located.

In order to find out the effect of spanwise wall movement on individual turbulence structures, including quasi-streamwise vortices, three simulations were carried out starting from identical fully turbulent initial conditions. One of the simulations was run maintaining the plane channel flow boundary conditions while the other two were run with the wall moved in each direction. This type of study makes it possible to see how individual structures behave in response to the skewing, the importance of the sign of the skewing and how the structures behave if there is no skewing applied. Since the streaks were observed to be broken up at $t = 0.3$, the behaviour of the structures at this time will be shown.

Figure 7 shows vector plots of velocity fluctuations ($u'_2$, $u'_3$) of the same flow-field in an undeformed state and deformed by movement of the walls in either spanwise direction. In all cases the mean flow is into the page and only a subset of the computational domain is shown. Unbroken lines correspond to contours of $p' = -2.8$ and broken lines correspond to contours of $u'_i = -2.5$. Most vortices can be observed by looking at low pressure surfaces while low speed streaks (which
Figure 7. Vector plots for plane channel flow (a), and deformed by skewing the wall in either direction, (b) and (c). The arrows are $u'_2$, $u'_4$ vectors, the lines are negative $p'$ contours and the dotted line show negative $u'_4$ contours. The spanwise mean velocity profile, $\overline{u_3}$, is shown on the right hand side of each plot. The streamwise flow direction is into the page.
are regions where the fluid is moving slower than the mean flow at that distance above the wall) can be observed by looking at surfaces where $u'_i < 0$.

Before examining the separate flowfields in detail, there are several important general points. Firstly, effects of the deformations are insignificant above $x_2^+ = 100$. Secondly, both directions of skewing have distorted the low speed streak structure. The skew flows then show formation of new streaks, which are oriented according to the sign of the skewing.

The plane channel vector plot (Figure 7a) shows a strong quasi-streamwise vortex with clockwise rotation and a vortex core at approximately $x_2^+ = 20$ and $x_3^+ = 160$. This vortex acts to eject slow moving fluid from the wall and this can be seen by the low speed streaks enveloping the ejection side of the vortex.

The wall movement has a significant effect on the behaviour of this quasi-streamwise vortex and its interaction with the streak. Wall movement from left to right (Figure 7b) shows that this vortex (now at $x_2^+ = 20$, $x_3^+ = 190$) has been weakened and the slow moving fluid is not ejected so much. The skewing has however enhanced an anti-clockwise vortex on the left of the original vortex. The liftup effect of this vortex forces slow moving fluid to join with the streak that is already present. Wall movement from right to left (Figure 7c) has swept the main vortex towards the wall and convected it in the skewed direction so that its core is at $x_2^+ = 105$ and $x_3^+ = 15$. Since the wall movement is in the same sense as the rotation on the lower side of the vortex, the ejection strength of the vortex is weaker. This is due to fluid particles being convected away as they pass through the lifting side of the vortex. This follows from the proposition of Eaton [8] for case 1 vortices shown schematically in Figure 1. The vortex created to the left of the main vortex in the previous case is not seen here.

The effect of wall movement on streaks and quasi-streamwise vortices depends on several factors such as the distance from the wall and the presence of other interacting structures. As a result of this study a mechanism for the behaviour of quasi-streamwise vortices in this flow is proposed for the region below $x_2^+ = 40$. The quasi-streamwise vortices, whose near wall velocity opposes the wall movement get destroyed by negation of the lower velocity while those whose rotation acts with the wall movement get swept towards the wall with reduced vortex size. The streaks are associated with the ejection sides of the vortices and thus wall movement of either sign reduces the coherence of existing organised streaks. This explanation for the behaviour of the near wall streak structures is different from that of Sendstad and Moin [31]. They state that the streak disruption is caused by convection of the quasi-streamwise vortices relative to the streaks whereas here we observe that both the quasi-streamwise vortices and the streaks are convected together and it is the rotational behaviour of the skewing on the quasi-streamwise vortices that affects the streaks since they are the principal generators of streaks. The explanation is also different from that of Coleman et al., as here we propose a mechanism for the behaviour of the vortices which was not included in that study. Further from the wall the wall movement has a less direct effect. The
structures respond to the behaviour of fluid below them so streak disruption can allow quasi-streamwise vortices of either sign to become larger. Thus the behaviour of quasi-streamwise vortices above $x^+_2 = 50$ is a lot less predictable. Since the main turbulence production and redistribution events occur below $x^+_2 = 50$, the reduction in vortex size and disruption of streaks below $x^+_2 = 50$ has a significant effect on the energy of the flow producing the reduction observed in Figure 5.

5. Turbulence Models Investigated

Standard Reynolds averaged closure methods are (i) two-equation linear eddy viscosity models, (ii) algebraic Reynolds stress and other non-linear eddy viscosity closures and (iii) full Reynolds stress (second moment) closures. In this study we focus on models in class (i), which are widely used in practice. Eddy viscosity modelling developed from an idea first suggested by Boussinesq in 1877. This can be written as

$$\overline{u'_i u'_j} = 2/3 K \delta_{ij} - v_t \left( \frac{\partial \overline{u'_i}}{\partial x_j} + \frac{\partial \overline{u'_j}}{\partial x_i} \right),$$

where the eddy viscosity, $v_t$, is

$$v_t = c_{\mu} f_{\mu} K^{(\frac{1}{2} - \frac{m}{n})} z^{\frac{1}{n}},$$

with $c_{\mu}$ as a constant of proportionality, $f_{\mu}$ a damping function, $K$ the turbulent kinetic energy and $z$ a closure variable. The main constraint is that the scalar variable, $z$, must have dimensions $L^{(\frac{n+2m}{n})} T^{-2m}$. The most popular choice is $m = 3/2$ and $n = -1$ making the new variable have the same dimensions as the dissipation rate of turbulence kinetic energy, allowing one to take $z = \varepsilon$. Alternatively some function of the dissipation combined with $K$, $z = z(K, \varepsilon),$ can be used. An example is $\omega \propto \varepsilon/K, (m = -1/2, n = -1)$ which has been used extensively [32]. The potential benefits and problems of this formulation will be discussed later. By solving transport equations for $K$ and $z$, the eddy viscosity can be found. This type of turbulence modelling is called two-equation modelling because it requires these two extra transport equations to be modelled in addition to the Reynolds-averaged Navier–Stokes equations.

The transport equation for $K$ is based on the exact transport equation for $K$ but is modelled using $\overline{u'_i}, K$ and $\varepsilon$ only. A common form for the $K$ equation is

$$\frac{D K}{D t} = P - \varepsilon + \frac{\partial}{\partial x_j} \left[ (v + \frac{v_t}{\sigma_K}) \frac{\partial K}{\partial x_j} \right] + \pi_K,$$
where \( P \equiv -u'_i u'_j \partial u'_i / \partial x_j \), \( \pi_K \) may be used to model the pressure diffusion in terms of \( K \) and \( \varepsilon \) while \( \sigma_K \) is normally a constant. The \( \varepsilon \) equation is much more loosely based on the exact dissipation equation. A common form is

\[
\frac{D\varepsilon}{Dt} = C_{\varepsilon 1} P \frac{\varepsilon}{K} - C_{\varepsilon 2} f_{\varepsilon} \frac{\varepsilon^2}{K} + \frac{\partial}{\partial x_j} \left[ (\nu + \frac{\nu_t}{\sigma_\varepsilon}) \frac{\partial \varepsilon}{\partial x_j} \right] + E + \pi_\varepsilon.
\] (8)

Here, \( C_{\varepsilon 1}, C_{\varepsilon 2} \) and \( \sigma_\varepsilon \) are normally constants, \( f_{\varepsilon} \) is a damping function while \( E \) and \( \pi_\varepsilon \) are further closure functions. Solution of the dissipation equation requires a boundary value for dissipation, which is not zero at the walls. There are three solutions to this problem: (i) set a wall value for \( \varepsilon \), (ii) solve a transport equation for \( \varepsilon \) (the “isotropic” dissipation), where \( \varepsilon = 0 \) at \( x_2 = 0 \), and add a non-isotropic part \( (D = \varepsilon - \varepsilon) \) to close the \( K \) transport equation, or (iii) combine transport equations of \( K \) and \( \varepsilon \) such that the new variable requires a zero boundary condition, i.e. \( z(K, \varepsilon) = 0 \) at \( x_2 = 0 \).

A parameter that has potentially attractive near-wall properties is

\[ g = \sqrt{K/\varepsilon}, \] (9)

which was proposed in a paper by Kalitzin et al. [16]. This has the benefit that it is zero at a wall. It has a potential problem that outside a boundary layer, \( g \) may become large and ad hoc corrections are needed. Because of the relatively simple boundary conditions required by \( g \) it has been studied in this work. The transport equation for \( g \) is obtained by direct transform of the Wilcox \( \omega \) equation

\[
\frac{Dg}{Dt} = -P \frac{5g}{18K} + \frac{0.075}{2c_\mu g} \frac{\partial}{\partial x_2} \left[ (\nu + \frac{\nu_t}{\sigma_\varepsilon}) \frac{\partial g}{\partial x_2} \right] - \left( v + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{3}{g} \left( \frac{\partial g}{\partial x_2} \right)^2. \] (10)

Durbin [7] has produced a different approach to modelling the eddy viscosity relation. He proposed modelling the eddy viscosity as \( \nu_t = c_\mu \nu^{2/3} T \), where the turbulent time scale is \( T = K/\varepsilon \) (except very close to the wall where \( K/\varepsilon < 6(\nu/\varepsilon)^{1/2} \) and the Durbin model makes use of a switch based on \( T = \max(K/\varepsilon, 6(\nu/\varepsilon)^{1/2}) \)) and \( \nu^{2/3} \equiv \overline{u'_2 u'_2} \) for plane channel flow. This involves solving an additional transport equation for the extra term, \( \nu^{2/3} \). This transport equation was developed based on the exact transport equation for \( u'_2 u'_2 \) and includes an elliptic relaxation model. For further details of the method, see [7, 19]. One criticism of this approach is that it cannot be used to model complex flows where there are interactions between the normal stresses since only one of the normal stresses is modelled. A possible reconciliation of this problem of the \( K - \varepsilon - \nu^{2/3} \) model for these flows is that the \( \nu^{2/3} \) equation no longer represents the actual \( \overline{u'_2 u'_2} \) of two-dimensional flow but rather a transport equation for the turbulence where this quantity is close to the actual \( \overline{u'_2 u'_2} \) in the near wall region. The \( K \) and \( \varepsilon \) transport equations have no significant modifications. Rodi and Mansour [26] carried out some a priori analysis of the Durbin [7] eddy viscosity relation to assess whether \( c_\mu \) was a constant for this relation and they found that for \( x_2^+ < 30 \) there was a minimum followed by a
Table I. Model constants and functions.

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<tr>
<td>( D )</td>
<td>( 2v \left( \frac{\partial \overline{\nu}}{\partial x_2} \right)^2 )</td>
<td>( 2v \frac{K}{x_2^2} )</td>
<td>( 2v \left( \frac{\partial \overline{\nu}}{\partial y_j} \right)^2 )</td>
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<td>( c_{\mu} )</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
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<tr>
<td>( C_{e1} )</td>
<td>1.44</td>
<td>1.35</td>
<td>1.44</td>
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<tr>
<td>( C_{e2} )</td>
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<td>1.92</td>
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<td>( \sigma_K )</td>
<td>1.0</td>
<td>1.0</td>
<td>( 1 - 0.5 \exp \left[ -\left( \frac{\overline{x}_2}{20} \right)^2 \right] )</td>
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<td>( \sigma_x )</td>
<td>1.3</td>
<td>1.3</td>
<td>( 1 - 0.5 \exp \left[ -\left( \frac{\overline{x}_2}{20} \right)^2 \right] )</td>
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<tr>
<td>( E )</td>
<td>( 2v\nu \left( \frac{\partial \overline{\nu}}{\partial x_2} \right)^2 )</td>
<td>( -2v \frac{x_2^2}{x_2^2} \exp \left[ -0.5x_2^+ \right] )</td>
<td>( 0.6\nu v \left( \frac{\partial \overline{\nu}}{\partial x_2} \right)^2 )</td>
</tr>
<tr>
<td>( \pi_e )</td>
<td>–</td>
<td>–</td>
<td>( -\overline{\nu} \frac{D}{\kappa} )</td>
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<tr>
<td>( f_{\mu} )</td>
<td>( \exp \left[ -\frac{3.4}{1 + \frac{2x_2}{20}} \right] )</td>
<td>( 1 - \exp \left[ -0.0115x_2^+ \right] )</td>
<td>( 1 - \exp \left[ -\frac{3x_2^+}{415} \left( \frac{x_2^+}{36} \right)^3 \right] )</td>
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<tr>
<td>( f_x )</td>
<td>( 1 - 0.3 \exp \left[ -\text{Re}_t^2 \right] )</td>
<td>( 1 - 0.22 \exp \left[ -\frac{\text{Re}_t^2}{6} \right] )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi_K )</td>
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<td>0</td>
<td>( -0.5v \left( \frac{\partial \overline{\nu}}{\partial x_2} \right)^2 \overline{D}^2 \kappa )</td>
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Figure 8. A priori comparison of the eddy viscosity relation for the Durbin [7] expression, the exact \( (\nu_t = -u_i \frac{\partial u_j}{\partial x_2}) \) DNS result and the standard model \( (\nu_t = 0.09K^2/\varepsilon) \) expression for steady channel flow.
tendency for \( c_{\mu} \) to asymptote to infinity. They also pointed out that very close to the wall the Durbin [7] eddy viscosity behaves as \( \nu_t \propto x_2^4 \) where the correct distribution is \( \nu_t \propto x_2^3 \) (although it should be noted that the standard \( k - \varepsilon \) model will also not give this asymptotic behaviour without modification). However the model has proved useful in many applications (e.g. [23]). Table I shows the constants and functions for some of the models that will be investigated in the current study. Figure 8 shows the a priori profiles for the basic eddy viscosity relation and the \( K-e-v^{2/3} \) relation as compared to the exact profile. This figure shows that the \( K-e-v^{2/3} \) model does not have the near wall peak associated with the \( K-e \) model and thus should not require near wall damping functions (which the \( K-e \) models rely on for this flow). There is some difference between the \( K-e-v^{2/3} \) solution and the exact solution further from the wall. The effect of adjusting the constant \( c_{\mu} = 0.2 \) to \( c_{\mu} = 0.25 \) is also shown. This indicates that this relation provides a significant improvement to the \( K-e \) type of model by removing the requirement for damping functions in the near wall region.

Model calculations are carried out using 81 points in the wall-normal direction on a stretched grid (approximately the same grid distribution as the DNS). Second order central differences are used for spatial discretisation and fourth order Runge-Kutta for time discretisation. The time step size is fixed to \( \Delta t = 0.001 \), which is the same as that used for the DNS. Mean velocity and turbulence kinetic energy profiles are shown in Figure 9 comparing model predictions with DNS. There are significant discrepancies between the DNS and the models even in the mean flow. Careful adjustment of the model constants can correct this so that the models have more accurate centreline velocities. However, these changes can affect other quantities in the modelling. The Kawamura and Kawashima [17] model seems to get the most accurate \( K \) profile. The Durbin [7] model has the peak in the right place but the wrong magnitude. The remaining models all predict incorrect peak locations and magnitudes. The \( K-g \) model for example gives a maximum value approximately half the DNS value. Figure 10 shows the turbulence production and dissipation for the models. This indicates that there are significant problems in modelling the dissipation near the wall despite the accurate production profiles. The shear stress, \( \overline{u_1' u_2'} \) (not shown), is modelled to a similar accuracy as the mean profiles. This is clearly expected given the close relationship between the shear stress and the mean velocity profile for parallel flows. It is not usually of primary importance to get accurate turbulence kinetic energy or dissipation profiles. However, large errors indicate inconsistencies in the modelling.

6. Modelling the Skew Flow

Since two directions are homogeneous and the models are derived using the Reynolds-averaged Navier–Stokes equations, these directions do not need to be modelled. Thus, for modelling purposes, the three-dimensional flow reduces to a one-dimensional problem. This has the benefit that the geometry is very simple
but the turbulence is complex. In this section performance of two-equation models will be assessed. Statistical effects, such as drops in turbulence intensity $K$ and skin friction $c_f$ should appear if the models are accurate.

Individual velocity profiles will be considered for each model at $t = 0.3$. Figure 11 shows the streamwise and spanwise mean velocity profiles for the models compared with the DNS. These plots show that the models are capable of calculating the mean flow to a similar accuracy to that of the original steady channel flow shown in Figure 9a.

A comparison of the turbulence kinetic energy profiles between the various models and the DNS is shown in Figure 12. This figure indicates that the profiles
of the Launder and Sharma [21] and $K-g$ models seem to match the DNS most closely. However, it is perhaps more accurate to say the turbulence has reduced to levels comparable to those predicted by these two models. Comparing these profiles to the profiles of steady channel flow shown in Figure 9b, it can be seen that the profiles have not significantly changed in magnitude. The relative change of the turbulence kinetic energy with time during the skewing can be investigated. The method of assessing the change was to examine how the maximum value of the turbulence kinetic energy varied. Since each model has a different profile for the turbulence kinetic energy, the comparison was made by normalising the values with the maximum value of the turbulence kinetic energy at the steady channel flow start condition. Figure 13 shows the behaviour of the turbulence kinetic energy.
Figure 11. A posteriori solutions showing (a) the mean streamwise and (b) the mean spanwise velocities in the skewed flow at $t = 0.3$. The two-equation models shown are Launder and Sharma [21] (LS), Chien [3] (CH), Kawamura and Kawashima [17] (KK), $K-g$ and Durbin [7] (DURB). These are compared with the velocity simulated by the DNS.

It was found that the cause of the drop in turbulence kinetic energy in this model relative to the others was the inclusion of the term

$$2
\nu
\frac{\partial^2 u_i}{\partial x_j \partial x_j} \frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

(denoted $E_{L,S}$) in the $\varepsilon$ equation. This term has been included in additional calculations using models of Chien (1982), $K-g$ and Durbin [7] and the resulting effect for
Figure 12. A posteriori solutions showing the turbulence kinetic energy in the skewed flow at $t = 0.3$. For key to line styles refer to the caption of Figure 11.

Figure 13. A posteriori solutions showing the change in the maximum turbulence kinetic energy with time during the skewing. $K_0\max$ is the maximum value at the steady channel flow state. The two-equation models shown are Launder and Sharma [21] (LS), Chien [3] (CH), Kawamura and Kawashima [17] (KK), $K-g$ and Durbin [7] (DURB). These are compared with the turbulence kinetic energy maxima simulated by the DNS.

skewed flow are shown in Figure 14a, with the models labelled respectively as CH 1, $K-g$ 1 and DURB 1 to indicate this modification. A different modification was applied to the Kawamura and Kawashima [17] model, KK 1, as will be discussed below. After inclusion of the extra term, each model was run to a steady state before applying the skewing in order to avoid changes due to a different balance of the equations. The effect of the modification is to reduce or slow the growth of the turbulence kinetic energy compared to the original models. This effect does not become significant until around $t = 0.1$. Thus the Chien [3] model (CH 1) still shows a large increase in turbulence kinetic energy in the early stages. At later
Figure 14. A posteriori solutions showing the change in the maximum turbulence kinetic energy with time during the skewing. For caption refer to Figure 13. The modifications in (b) are based on the proposal of Rodi and Mansour [26] for the term $p_3$. For further details refer to Section 6.

stages in time, the effect of the modification becomes more apparent. The Chien [3] model shows a significant drop from its highest level but does not fall below its initial steady flow level. The main difference between this model, Durbin [7] and $K-g$ is the use of damping functions based on $x_2^+$ in the eddy viscosity and dissipation equation (see Table II). This implies that these functions are contaminating the model in the skewed flow. This hypothesis can be checked by replacing $x_2^+$ with $0.6\text{Re}_y$, where $\text{Re}_y = x_2\sqrt{K}/\nu$, in the Chien [3] model. This change was made and combined with the inclusion of the additional term $E_{LS}$. The result for this is
also shown in Figure 14a (labelled CH 2). This change has now made the initial behaviour of this model much closer to the DNS. The Kawamura and Kawashima [17] model was modified in the same way and is also shown in Figure 14a (labelled as KK 1). This modification prevents the rapid increase in turbulence kinetic energy in response to the skewing observed in the original model. The Durbin [7] and K–g models both show reductions in turbulence kinetic energy in a similar way to the Launder and Sharma [21] model although the minima occur at different points in time.

Rodi and Mansour [26] carried out some detailed analysis of the entire dissipation transport equation and found that the term $E_{L5}$ was connected to one of the terms in the exact transport equation

$$P_3^e = -\nu \frac{\partial u'_i u'_j}{\partial x_m} \frac{\partial}{\partial x_k} \left( \frac{\partial u_i}{\partial x_m} \right).$$  \hspace{1cm} (11)

They identified the primary source terms for plane channel flow in the transport equation of $(\partial u'_k u'_i / \partial x_m)$ to be $-(\partial u'_k u'_i / \partial x_2) \partial u_1 / \partial x_2$ and $-u'_2 u'_2 \partial^2 u_1 / \partial x_2^2$. In order to use this type of expression in a K–ε model they then converted the normal stress $u'_2 u'_2$ to $K$ and multiplied the terms by the turbulent time scale $K/\varepsilon$. Thus, the proposed model for this term was

$$P_3^e = C_1^3 2 \nu u'_2 \left( \frac{\partial^2 u_1}{\partial x_2^2} \right)^2 + C_2^3 2 \nu \frac{K \partial K}{\varepsilon} \frac{\partial u_1}{\partial x_2} \frac{\partial^2 u_1}{\partial x_2^2}$$  \hspace{1cm} (12)

for shear flows with $C_1^3 = 0.5$ and $C_2^3 = 0.006$. A priori tests of the model expression for this term, carried out by Rodi and Mansour [26], showed it to be very close to the exact term in the dissipation equation. Since the Durbin model makes use of the $u'_2 u'_2$ normal stress explicitly, a test was carried out to use the original form for the source terms of the $P_3^e$ equation. This test demonstrated that this form could significantly improve the model behaviour for this flow. The expression for the extra term was

$$P_3^e = 1.3\nu T \left[ \frac{u'_2 u'_2}{\partial x_2^2} \left( \frac{\partial^2 u_1}{\partial x_2^2} \right) + \frac{1}{2} \frac{\partial (u'_2 u'_2)}{\partial x_2} \frac{\partial u_1}{\partial x_2} \frac{\partial^2 u_1}{\partial x_2^2} \right],$$  \hspace{1cm} (13)
and the results produced are shown in Figure 14b (labelled DURB 2). Also shown in this figure are modified models of Kawamura and Kawashima (KK 2) and \(K-g\) (\(K-g\) 2). For the former the modification was to convert the \(x_2^+\) damping function to \(2.4K^2/(\nu\varepsilon)\) and include the additional term

\[
P_e^3 = 3
\nu
\left( \frac{\partial^2 u_1}{\partial x_2^+} \right)^2 + 0.002\nu \frac{K \partial K}{\varepsilon} \frac{\partial u_1}{\partial x_2^+} \frac{\partial^2 u_1}{\partial x_2^+}. \tag{14}
\]
Figure 16. A posteriori solutions showing (a) the mean streamwise and (b) spanwise velocities in the skewed flow at \( t = 0.3 \). The modified two-equation models shown are Kawamura and Kawashima [17] (KK 2), \( K-g \) (K–g 2) and Durbin [7] (DURB 2). These are compared with the velocity simulated by the DNS.

For the \( K-g \) model the modification to the \( g \) equation was the inclusion of

\[
P_e^3 = 3 \nu \nu_l \left( \frac{\partial^2 \bar{u}_1}{\partial x_2^2} \right) - 0.1 \nu g^2 \frac{\partial K}{\partial x_2} \frac{\partial^2 \bar{u}_1}{\partial x_2^2} \frac{\partial^2 \bar{u}_1}{\partial x_2^2}
\]

and use of \( \sigma_e = 0.3 \) and \( \alpha = 2/5 \). The constants have been optimised for each model separately. These results demonstrate that the Rodi and Mansour addition improves all the models for the skew flow and the Durbin model, which is most closely connected to the primary source terms, then gives the most accurate solution. This type of model development tends to contradict some observations made...
by Wilcox [32] and Behnia et al. [1] that the dissipation transport equation should
not be compared closely with the exact dissipation transport equation.

Although the models all show significant improvement in their behaviour for
this flow there remains the problem that the centreline velocities and turbulent
kinetic energy peaks (Table II) are now changed, indicating that other parts of
the model also need improving, or constants re-tuning. In order to get a clearer
picture of the effect of the modifications the velocity and turbulence kinetic energy
profiles for steady flow and the skewed flow are plotted in Figures 15, 16 and
17. The most significant change, as indicated in Table II is the reduction in the
turbulence kinetic energy of the modified Kawamura and Kawashima [17] model
in plane channel flow. The profiles for the modified models in the skewed flow
show that the modifications have improved overall behaviour for this flow.

Sarghini et al. [29] carried out a large-eddy simulation (LES) study of a similar
flow (in which only one wall was skewed). In general it can be seen that the LES
results [29, fig. 10(b)] for the drop in the turbulence kinetic energy are closer to the
DNS than those shown here for the standard two-equation models. However, the
improvement applied to the Durbin model (Figure 14b) shows a result which seems
to be as good as some of the better LES results, even though the modification has
changed the steady flow values as shown in Table II and Figure 15. The level of
error in the mean velocity is in the same range as that obtained for different LES
models as shown in [29, fig. 3(a)]. This implies that the turbulence involved in this
type of flow can in principle be modelled quite well at the level of two-equation
modelling without the need to resort to higher order closures or methods such as
LES.
7. Conclusions

Both direct numerical simulation and modelling of a time-dependent skewed turbulent channel flow have been investigated. Instantaneous DNS flowfields involving skewing in both spanwise directions showed that the near wall quasi-streamwise vortices to be reduced in both strength and distance from wall due to the skewing. The mechanism for this is close to that proposed by Eaton [8]. Vortices whose rotation was the same sign as the skewing were convected in the direction of wall movement and swept towards the wall while vortices with opposite sign with respect to the skewing were significantly disrupted. In both cases the lifting side of the vortices was affected most. The time at which the disruption to the turbulence was largest corresponded to the time at which the mean quantities such as turbulence kinetic energy reached their minimum values. Two equation turbulence modelling of this flow did not capture this significant drop in mean turbulent kinetic energy. Models which used wall distance $x^+$ in their damping function could be improved significantly by converting these terms to a $Re_y$ dependence. An improvement based on the exact dissipation transport equation was also investigated. The modification was found to improve all the turbulence models for this flow and allow a drop in the turbulent kinetic energy to occur. The modified $K-e-u'^2$ model was found to provide a result that was very close to the result predicted by the DNS. A possible reason for this is that the improvement is initially based on the turbulence component, $u'^2$, which can be recovered exactly for this model, whereas other models require this quantity to be approximated by $2/3K$.

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