A STRESS-STRAIN LAG EDDY VISCOSITY MODEL FOR MEAN UNSTEADY TURBULENT FLOWS

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Abstract

This thesis describes the development and validation of an Eddy Viscosity Model (EVM) that is sensitive to the local stress-strain mis-alignment of mean unsteady turbulent flow, in the framework of Unsteady Reynolds Averaged Navier-Stokes (URANS) turbulence models. The new model considers a parameter, $C_{as}$, representing the dot product of the strain tensor $S_{ij}$ and the stress anisotropy tensor $a_{ij}$. This quantity, defined as $C_{as} = -a_{ij}S_{ij}/\sqrt{2S_{ij}S_{ij}}$, projects the six equations of the Reynolds stress transport model onto a single equation. With respect to Non-Linear Eddy Viscosity Models (NLEVM) or Explicit Algebraic Reynolds Stress Models (EARSM), the novelty of the present model lies in incorporating time dependent and/or transport effects of the turbulent stresses.

Initially, simple time dependent homogeneous flows are analysed, which are of relevance to the unsteady turbulent flow in the cylinder of a piston engine. Oscillating Channel flows are then examined, to assess the model’s performance in the case where turbulence is most influential near to a wall. The model is then applied to two-dimensional flows of direct industrial relevance: the unsteady turbulent flow around an aerofoil beyond stall, and the complex unsteady turbulent motion in the wake of a cylinder. The later case is also calculated for a fully three-dimensional domain. The $C_{as}$ model is incorporated into the widely used Shear Stress Transport (SST) model to form a three-equation model, and good predictions are returned with the new model, relative to the standard SST model, for many aspects of the flows presented. The small additional computational expense is measured to be around 10 – 15% compared to the standard SST model.

The $C_{as}$ model is shown to be a simple and economic modification to an existing modelling scheme that is widely used in industry. While this work represents only the first stage of the development and optimisation process, the general conclusions of this thesis are encouraging enough to warrant further investigation. Observations from the 3D calculations suggest that this model might also be exploited in a Detached Eddy Simulation (DES) framework, or as part of a hybrid RANS-LES approach.
Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or institute of learning.
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This work is dedicated to Cameron, my little brother,
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and to my Parents.
Chapter 1

Introduction

1.1 Background

Computational Fluid Dynamics (CFD) has risen from a mathematical idea to become an integral tool in almost every branch of fluid dynamics, from weather prediction through vehicle performance to biological analysis. CFD generally refers to the numerical solution, by computational methods, of the governing equations which describe fluid flow.

As a developing science, Computational Fluid Dynamics has received extensive attention across the international community since the advent of the computer. The subject has matured considerably over the past four decades, a rapid ascent in tandem with the exponential growth of computational power in the same period. More memory and faster calculation speed allow the user to use more complex models with fewer approximations, as well as a higher resolution in both space and time. Progress has been driven by the desire to model physical phenomena that cannot be easily measured with a physical experiment, and by the cost and time benefits it offers over those cases where experiment is feasible.

In CFD, numerical algorithms are used to reach an approximate solution to the flow field, which is commonly represented by a discrete set of nodes, defined by a mesh specifically tailored to the geometry of the problem. The variation of physical values across the flow field can be expressed exactly by the differential Navier-Stokes equations which, in CFD, are replaced with sets of algebraic expressions known as discretised equations. Thus, instead of a closed-form analytical solution, the end product of CFD is a collection of numbers at discrete space and time locations.

In design and development, CFD programs are now considered to be standard numeri-
cal tools, widely utilised within industry, where commercial CFD codes potentially allow a non specialist user to make predictions of almost any flow. As a consequence there is a constant demand for improved accuracy in progressively more complex cases. The vast majority of industrially relevant flows involve turbulent flow to some extent, and as such, CFD requires a means for the inclusion of the effects of turbulence. Turbulence has earned itself a bad reputation because of its highly complex and elusive nature, with one Nobel prize winning physicist referring to it as “the most important unsolved problem of classical physics”\(^1\).

Turbulence is composed of eddies: patches of swirling fluid, moving seemingly randomly around the mean direction of motion. This unsteady state of fluid motion arises when the inertia of the fluid exceeds a specific threshold, below which viscous forces damp out this chaotic behaviour. This threshold is generally characterised by the Reynolds number which is a non-dimensional measure of the ratio of inertial to viscous forces.

The difficulty in predicting turbulence arises from the nonlinearity of the Navier-Stokes equations, which generate a broad range of length and time scales, with several orders of magnitude between the smallest and the largest eddies. For example, a typical 50\(m\) long transport airplane cruising at 250\(ms^{-1}\), would require of the order of \(10^{16}\) grid points to simulate the turbulence near the surface with reasonable detail. Even with the most powerful supercomputer currently available it would take thousands of years to compute the flow for one second of flight time! However, in most cases it is not necessary to resolve all scales since the largest eddies present in the flow account for the majority of turbulent motion. It is often proposed that these coherent structures, of scales similar to the characteristic scales of the flow, may account for around only 10\% of the turbulent energy, but are responsible for as much as 90\% of the transport of flow properties (Ferziger, 2002).

In light of this observation various different approaches have been developed and applied, with a hierarchy of complexity, which can be broadly categorised into three groups: Direct Numerical Simulation (DNS), Large Eddy Simulation (LES) and Reynolds Averaged Navier Stokes models (RANS). Of these only Direct Numerical Simulation offers a complete resolution of the flow physics without resorting to modelling, directly applying the discretized Navier Stokes equations onto a numerical domain with sufficiently fine grid resolution so as to account for the smallest scales. This method is extremely expen-

\(^1\)Richard Feynmann 1918-1988
sive and as such is limited to applications of academic interest, but it remains a valuable tool for providing fundamental insights into the basic mechanisms of turbulence as well as reference data for numerical models in simple flows.

Early studies on homogenous turbulence were initiated by the work of Orsag and Patterson (1972) on spectral methods, and by work on numerical algorithms by Rogallo (1981). In addition to isotropic homogenous turbulence, a variety of imposed mean velocity gradients were investigated (Rogallo and Moin, 1984), as well as low-Reynolds-number wall bounded flows (Kim and Moin, 1987), and the resulting data sets from these works have been used extensively. More recently, channel flows at higher Reynolds numbers have been simulated and large databases have been compiled to provide modellers with a broad range of data (AGARD, 1997). Recently, advances in CFD and computational power alike have allowed DNS studies of more complex flows to be made, such as the flow through turbine blades (Kalitzin et al., 2003). However, at high Reynolds numbers it soon becomes impractical to resolve all the scales of motion, especially since the relative influence of the smallest eddies on the larger dominant structures becomes much less important, and as such DNS is likely to remain predominantly an academic tool.

The pioneering work on Large Eddy Simulation by Smagorinsky (1963) and Deardorff (1974) amongst others, was motivated by meteorological applications where the large-scale motion is of particular interest. Only the largest eddies are simulated in their entirety while the smaller-scale eddies are merely modelled. The velocities are decomposed into filtered and modelled components, with the resolution of the numerical grid determining the filter size, and the modelled quantities referred to as the sub grid scale motion. The filtered Navier-Stokes equations resolve the largest scales of motion and as such are expected to provide an accurate prediction of the flow whilst requiring less computational resources than DNS. There has been a considerable amount of work in this area in the last decade and the technique has been used successfully on a range of industrial applications, eg. a swirling coaxial jet combustor (Lin, 1998).

If instantaneous (or filtered) flow parameters are not required, the main flow equations can be averaged to obtain what are known as the Reynolds Averaged Navier Stokes (RANS) equations, allowing mean flow parameters to be modelled without the need to know the detail of the turbulence at every moment in time. These equations result from the decomposition of the velocity vector into mean and fluctuating parts, where the mean is defined as the average of the instantaneous velocity. In the averaging process, informa-
tion regarding the turbulent fluctuations is lost, which appears in the RANS equations in the form of correlations of fluctuating velocities known as Reynolds stresses, $\overline{u_i u_j}$. These quantities appear explicitly in the momentum equations and require modelling in some way to close the set of equations. This is the objective of turbulence modelling and is often done by relating $\overline{u_i u_j}$ to the mean flow through an algebraic relationship derived from an analogy with the stress-strain relationship in solid mechanics.

Eddy viscosity models (EVM), use the hypothesis that turbulent eddies act on the flow in the same manner as molecular viscosity. A functional relationship is assumed between the stress and strain fields which is dependent upon the local nature of the turbulence in the form of the eddy viscosity at that point. The most popular models require two transport equations, in order to obtain an approximation of the turbulent length and time scales at that point. Many variations of these models exist, this area constituting the most consistently active area of modelling research across the past three decades. Modifications to this general form of model may employ either a single transport equation or a set of three, while others propose non-linear forms of the stress-strain relationship which include higher order terms that enable the stress field to be approximated with better accuracy. A more sophisticated approach, known as Reynolds stress transport modelling or simply Reynolds Stress Models (RSM), requires the solution of seven transport equations, one for each of the Reynolds stresses themselves, and an additional equation to obtain the lengthscale of the local turbulence. In more complex flows with features such as separation, recirculation, curvature and swirl, the stress field is highly anisotropic and the RSM approach is usually required to account accurately for these effects.

Without doubt, RANS models have been the most commonly used approach for predicting turbulent flows, and solving for only mean flow parameters is not a significant disadvantage since in the majority of cases engineers require only mean values as opposed to vast amounts of instantaneous data. Also, since these approaches are significantly less expensive to use in terms of computational power, they allow for solutions to be obtained within the demanding timeframes stipulated in industry: normally less than 24 hours. However, all RANS turbulence models are one point closures, meaning they are measures of the turbulent quantities at a single point in space and time. As such they are unable to account for so called non-local effects such as pressure-strain fluctuations and wall echo, which are two examples of how the pressure at a given location may be dependent upon a region of the flow far larger than a single point. A higher level of flow description is
possible when considering multi-point closures, but such approaches are highly complex and hence some of the advantages of Reynolds averaged closures over LES are reduced.

It has been predicted that LES calculations for complex geometries at high Reynolds numbers, for example the flow over an aircraft, may be out of reach for several decades to come (Spalart, 2000). As such, RANS models currently remain the standard industrial approach to treating turbulence in CFD, and research in this field continues to be influential. Unsteady RANS (URANS), attempts to account for the large scale unsteadiness of strongly three-dimensional flows with some success, and this has led to the conception of a series of new approaches which can be collectively referred to as Hybrid LES. In such methods, URANS models are used together with the Large Eddy Simulation approach, where the switch from one model approach to the other may be zonal, where geometric boundaries are predefined, or analytical, by use of a local turbulence lengthscale or similar. One such approach which has received a lot of attention in recent years is Detached Eddy Simulation (DES), whereby URANS is used in the near wall ‘attached’ flow regions while reverting to LES in the far field. This approach has yielded some interesting results, but considerable problems lie in the definition of different modelling regions in the flow, and at the interface between them. This is generally not too complicated for flows with a strong separation, for example flow past bluff bodies, but can become less obvious when there is little, or indeed no, separation in the time-averaged flow field.

Advances in turbulence modelling are clearly required to handle the ambitious calculations envisaged in industry. Given that the computational strengths of CFD are constantly increasing, turbulence modellers are beginning to focus on the examination of new approaches, to achieve a higher level of resolution despite the cost of higher computational resources. The precise nature of what is required remains unclear, but whether they are used alone, or coupled in a hybrid framework, advanced URANS models will have an important role to play in industrial CFD for the next few decades.

1.2 Study Objectives

Many everyday turbulent flows are inherently unsteady, both industrial applications (e.g. flow in the cylinders of internal combustion engine, turbines, flow in the wake of bluff bodies... ) and flows in nature (e.g. blood flow in arteries, air in lungs... ). The unsteadiness in these flows can be a result of imposed fluctuating boundary conditions, or geom-
etry induced oscillations, or a combination of both. The presence of such unsteadiness in a flow can significantly alter the evolution of important parameters such as Reynolds stresses $\overline{u_iu_j}$, turbulent kinetic energy $k$, and dissipation rate $\varepsilon$. Despite the existence and derivation of progressively more complex modelling, simulation or hybrid approaches in the field of turbulence modelling, URANS models are widely used for the computation of complex unsteady flows. The accuracy of standard eddy viscosity models has been reasonably well established and is known to be rather poor in some cases; several fundamental assumptions in their derivation for steady quasi-equilibrium flows no longer hold in the application to transient cases.

Despite their widespread use, two equation models possess several major flaws in their underlying theory which limits their success in the modelling of complex unsteady flows. Some of these issues are addressed with Reynolds stress models, but these models are more expensive to run as an additional five transport equations are required, and they are also prone to convergence problems unless special numerical treatments are adopted.

In the calculation of unsteady flows, the eddy viscosity calculated using a two equation model often differs greatly from the equivalent value implied by a Reynolds stress model. In general, two equation models fail to account correctly for the transport or history effects in unsteady mean flow; by assuming the normal stresses to be isotropic and calculating the transport of the combined mean turbulence energy, $k = 0.5 \overline{u_iu_i}$, any variation of the individual components is effectively ignored. There seems to be little doubt that modelling the transport of the stresses leads to a more realistic representation of turbulent flow; improved prediction of the Reynolds stresses will be reflected in the distribution of energy and in the transport of momentum. It is not uncommon for two equation models to predict a steady solution for flows that have mean unsteady motion, for example periodic vortex shedding. Many two equation models are known to over-predict the level of turbulence energy in regions of high strain, and fail to account for the effects of body forces such as rotation or buoyancy. As a result these solutions exhibit a larger diffusion of momentum, due to higher levels of eddy viscosity, which in turn tends to damp out any coherent structures in the flow.

The fundamental assumption in the development of many standard models is the hypothesis from Boussinesq, that there exists a property of a flow which dictates the stress-strain relation, the eddy viscosity, analogous to the molecular viscosity in laminar flow. This property, which depends on the flow conditions at a given position in space and time,
is used to approximate the Reynolds stresses from the local velocity gradients. For linear eddy viscosity models, this implies a constant co-linearity of the principal axes of the turbulent stress and mean strain tensors; a configuration that is unlikely to be found in the majority of flows, and one which is not even the case in simple homogenous shear flow.

Even if the linear stress-strain hypothesis is accepted, the tensor relating stress to strain should be of fourth rank. However, the eddy viscosity, $\nu_r$, is usually approximated from a product of local scales of the turbulence, with the structure parameter $C_\mu$ which generally takes a constant value of 0.09. This is a significant assumption, and one that is appropriate only for near equilibrium turbulence (where the ratio of production to dissipation of turbulence energy is close to unity) in simple shear. Even for simple flows in local equilibrium this value of $C_\mu$ remains a general approximation, and so when complex unsteady flows are to be modelled, various shortcomings can be expected. The Reynolds stress model provides a more coherent framework for the prediction of the turbulent stresses by considering the transport of each individual component, which goes beyond an explicit link to the mean velocity field. The motivation behind the present work is to quantify the relative alignment of the stress and strain tensors and to model the evolution of this parameter. The hope is that by incorporating this feature into the stress-strain relationship, some of the transport effects captured by using a full Reynolds stress transport model may be passed onto an eddy viscosity model.

It is the objective of this work to approximate the stress-strain missalignment by using a parameter that can be modelled locally in the flow field. This would lead naturally to the development of a transport equation to be used together with a two equation model, so as to bridge the gap to the full RSM approach.

### 1.3 Outline of Thesis

Chapter 2 is a review of the literature for RANS modelling, beginning with a description of the principal approaches for the closure of the RANS equations. The modelling issues that are of particular interest in the context of this work are then discussed in more detail. Chapter 3 summarises the finite volume method, and details of the code which is used for the calculations reported in this thesis. Chapter 4 then provides specific information of the implemented form of a number of existing turbulence modelling schemes that have been used for comparisons in the present work.
Chapter 5 presents the novel modelling approach that has been developed in this study, including its full derivation and numerical treatment.

Chapter 6 describes the performance of the new model for simple homogeneous shear flows, where both steady and unsteady cases are analysed. Chapter 7 describes the extension of the model to inhomogenous flow, with applications shown for fully developed channel flows. Chapter 8 then presents results for oscillating channel flows which are considered in order to enable an appraisal of the near wall performance of the new model.

Chapter 9 analyses the unsteady case of separated flow around a NACA0012 aerofoil beyond stall. Comparisons are made between existing models and the scheme proposed in this thesis. Chapter 10 presents results for the case of the flow over a cylinder in a square duct. First a two-dimensional assumption is made and then the full three-dimensional flow is calculated. A detailed discussion of the results is then made. Finally, Chapter 11 includes conclusions together with suggestions for future work.
Chapter 2

Literature Review

2.1 Turbulence Modelling

This thesis focuses on RANS models for incompressible flows and the analysis of their performance for unsteady mean flows. This chapter reviews existing such models in depth with particular attention paid to the developments relevant to this current work.

2.1.1 Navier-Stokes Equations

The instantaneous Navier-Stokes equations are derived from the principles of conservation of mass and momentum. These equations can be expressed for an incompressible Newtonian flow with a constant density and a variable viscosity in Cartesian form as follows with $x_j$ as the space coordinates, where $\Phi$ denotes an instantaneous quantity while $\rho$ is the density of the fluid and $\nu$ is its kinematic viscosity:

\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0 \quad (2.1)
\]

\[
\frac{\partial \tilde{U}_i}{\partial t} + U_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \frac{\partial^2 \tilde{U}_i}{\partial x_i \partial x_j} \quad (2.2)
\]

The same equations can be expressed in vector form as follows where $\Delta$ is the Laplacian operator, since for compressible flow $\Delta \tilde{U} = \nabla \cdot (\nabla \tilde{U} + \nabla \tilde{U})$:

\[
\nabla \cdot \tilde{U} = 0 \quad (2.3)
\]

\[
\frac{\partial \tilde{U}}{\partial t} + \nabla \cdot (\tilde{U} \otimes \tilde{U}) = -\frac{1}{\rho} \nabla \tilde{P} + \nu \Delta \tilde{U} \quad (2.4)
\]
Analytical solutions for these equations exist for a few simple laminar flows, such as the Poiseuille or the Couette flows\(^1\) amongst others. The equations can be solved directly as in DNS although this is costly and has limited industrial application as previously discussed. The instantaneous Navier-Stokes equations 2.1 and 2.2 are averaged to obtain the RANS equations by decomposing the dependent variables as \(\Phi_i = \Phi_i + \phi_i\) where \(\Phi_i\) is the mean component and \(\phi_i\) the fluctuating component, resulting in

\[
\frac{\partial U_i}{\partial x_i} = 0
\]

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left( \frac{\partial U_i}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left( u_i u_j \right)
\]

If the mean flow is stationary the first term in equation 2.6 is neglected and a steady solution can be obtained, unlike in DNS and LES where an unsteady solution is always obtained (although the solution may be statistically steady in time). As previously noted, information is lost in the averaging process which gives rise to the appearance of the Reynolds stress tensor \(u_i u_j\). It is the role of turbulence models to provide an approximations to these terms.

When using the RANS equations to calculate unsteady flows, in the case of Unsteady RANS (URANS), a little more care has to be taken regarding the averaging process. Instead of a simple time averaging, this should now be interpreted as an ensemble averaging. In the case of unsteady flows exhibiting a mean periodic oscillation about a long-time average (such as the unsteady flows considered in this thesis), this long-time average may be denoted by \(\Phi_t\). If the mean departures from this long-time average are denoted by \(\Phi'_t\), then the instantaneous velocity field may now be decomposed as \(\Phi_i = \Phi_t + \Phi'_t + \phi_i\), where \(\langle \Phi_i \rangle = \Phi_t + \Phi'_t\) is often referred to as the phase averaged velocity.

### 2.1.2 Eddy Viscosity Models

The basis for the vast majority of turbulence models stems from the Eddy Viscosity model (EVM) originally proposed by Bousinesq. He approached the closure problem by assuming the turbulent stresses to be proportional to the mean rate of strain, introducing an apparent viscosity as the proportionality term (Boussinesq, 1877):

\(^1\)The Poiseuille flow is the laminar flow in a circular pipe, and the Linear Couette flow refers to the laminar flow between two planes of constant separation, one of which is moving relative to the other.
\[ -\vec{u}_i \vec{u}_j = 2\nu S_{ij} - \frac{2}{3} \delta_{ij} k \]  \hspace{1cm} (2.7)

where \( \delta_{ij} \) is the Kronecker delta, \( k \) is the turbulent kinetic energy, defined as half the trace of \( \vec{u}_i \vec{u}_j \), and the mean strain rate tensor is

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \]  \hspace{1cm} (2.8)

In contrast to the molecular viscosity, \( \nu \), that is an intrinsic property of the fluid, the eddy viscosity, \( \nu_t \), depends upon the local flow conditions. The sum of the molecular and eddy-viscosity is usually known as the effective viscosity. Introduction of the effective viscosity into the Navier-Stokes equations in place of the molecular viscosity is convenient in both practical and numerical terms. It allows for a straightforward implementation of turbulence models into existing Navier-Stokes solvers and it also tends to enhance numerical stability since it introduces additional diffusion into the momentum equations (given the caveat that the effective viscosity is generally always positive). Therefore, solving the RANS equations is analogous to the solution of the Navier-Stokes equations for a laminar flow, but with a modified fluid viscosity which is the sum of the laminar and turbulent parts.

The Reynolds stress tensor is a three-by-three symmetric tensor (i.e. \( \vec{u}_i \vec{u}_j = \vec{u}_j \vec{u}_i \)) with an isotropic state of \( \overline{\vec{u}_1 \vec{u}_1} = \overline{\vec{u}_2 \vec{u}_2} = \overline{\vec{u}_3 \vec{u}_3} = 2/3k \). Deviation from isotropic values is usually characterised by the stress anisotropy tensor:

\[ a_{ij} = \frac{\overline{\vec{u}_i \vec{u}_j}}{k} - \frac{2}{3} \delta_{ij} \]  \hspace{1cm} (2.9)

The inherent assumption in the linear EVM is that this anisotropy is determined by the local velocity gradients, which is in fact only valid for one component of the tensor in a simple shear flow. However it is important to note that despite this major simplification, linear EVMs have been shown to perform in a variety of flows with reasonable accuracy.

The eddy viscosity is, by dimensional analysis, the product of a length scale and a velocity scale. Since it is not a bulk property of the fluid itself, but of the fluid’s motion, a suitable method of specifying the local variation of these scales is required. Hence the turbulence closure problem is reduced from approximating six Reynolds stresses to finding appropriate length and velocity scales.
Zero equation models

Purely algebraic models are generally the simplest of all turbulence models (also referred to as zero-equation models, referring to the number of transport equations solved). The eddy viscosity was approximated by Prandtl (1925) in terms of a mixing length that is analogous to the mean path in a gas (the kinetic theory of gases). This is known as the Mixing Length Hypothesis (MLH):

\[ \nu_t = l_m^2 \| S \| \]  

(2.10)

where the mixing length, \( l_m \) is an algebraic expression and \( \| S \| = \sqrt{2S_{ij}S_{ij}} \) is an invariant of the tensor \( S_{ij} \). Prescription of the mixing length depends upon the nature of the local flow. For example for the fully turbulent region of equilibrium boundary layers, \( l_m \) can be approximated using a simple linear relationship, \( l_m = \kappa y \); the product of the von Karman constant, \( \kappa = 0.41 \), and the wall distance (coordinate dependent but commonly denoted by \( y \)). This model has had a substantial influence on the development of turbulence modelling, offering both a simple modelling approach and a means for basic analytical interpretation. Various improvements upon this model have been proposed, for example, the model by Baldwin and Lomax (1978), which has been used successfully for years in aerospace applications where the turbulence is confined to a thin turbulent boundary-layer. However, considerable variations are required for more complex flows; in particular these models predict that the turbulent viscosity vanishes when the strain rate falls to zero which, for instance, presents a problem for separated boundary layers where the turbulence can indeed be significant.

Notable also is the modification proposed by Van Driest (1956) who redefined the mixing length with the inclusion of a damping function in order to account for a reduction in the length scale as a solid boundary is approached. Hence, the mixing length becomes \( l_m = \kappa y f \), where \( f \) is a damping function defined as:

\[ f = 1 - \exp \left( \frac{-y^+}{26} \right), \]  

(2.11)

\( y^+ = u_t y / \nu \) is the non-dimensional distance from the wall; \( u_t = \sqrt{\tau_w / \rho} \) is the friction velocity and \( \tau_w = \mu (dU/dy)_{wall} \) is the wall shear stress.
2.1. Turbulence Modelling

One equation models

Analysis of algebraic model performance motivated the development of transport equations for turbulent scales, with the inherent ability to convect and diffuse values in order to allow turbulence to be non zero in regions where source terms may be zero. The first of these such models, the so called one-equation model requires the solution of a transport equation for the turbulence kinetic energy, $k$, which is used to obtain the velocity scale for the turbulence instead of the mean velocity gradient, thereby eliminating the prediction of a zero eddy viscosity whenever the velocity gradient vanishes. This method was first proposed by Prandtl (1945), with the eddy viscosity defined as follows:

$$\nu_t = C_\mu k^{1/2} l$$ (2.12)

where $C_\mu$ is a constant and the length scale $l$ is prescribed algebraically similarly to in the zero equation models. Unfortunately, this model inherits the problem of defining an appropriate length scale: a geometry dependent prescription is required for each application.

A more recent one equation model proposed by Spalart and Allmaras (1992) overcomes the above problem by directly solving a transport equation for the eddy viscosity $\nu_t$. Earlier proposals for such a model are described by Nee and Kovasznay (1969) and Baldwin and Barth (1990). The Spalart and Allmaras model has been considerably successful and continues to be used for many aerospace applications, in particular in cases where the flow is known to be attached. Unfortunately the method does not perform as well for separated or recirculating flows where, similarly to previous approaches, the model suffers from the need to prescribe an appropriate turbulent length scale.

Two equation models

It became apparent that a more robust modelling approach would be obtained if the length scale were obtained without the need to refer to the distance from the wall. In light of this, a series of models evolved which solved transport equations for both the turbulent length scale and the velocity scale and, as such, these were denoted two-equation models. These have become without a doubt the most successfully used and widely applied models. It is a testament to academic engineering that such an elusive phenomenon as turbulent motion can be approximated by such rudimentary rules. Many two-equation models have
been proposed, which differ principally in the variable from which the length scale is derived. Kolmogorov (1942) proposed a model for the mean frequency of the most energetic eddies, \( f = k^{1/2}/L \); Rotta (1951) proposed transport equations for the integral length scale, \( L \), and the turbulent kinetic energy and length scale combined, \( kL \). Wilcox (1988) proposed a model with the turbulent length scale derived from a transport equation for the turbulent frequency, or more correctly, the dissipation rate per unit kinetic energy, \( \omega = k/l^2 \). This approach, following the logic originally put forward by Kolmogorov has been particularly popular. Wilcox (1994) discusses the performance of the \( k - \omega \) model, noting its good performance in two-dimensional boundary layers and recirculating flows, but highlighting its undesirable sensitivity to freestream boundary conditions in free or semi-confined shear flows. Perhaps the most widely used length scale determining variable, is the dissipation rate of turbulent kinetic energy, \( \varepsilon \). This quantity is the rate of dissipation of turbulent kinetic energy per unit mass, which appears explicitly in the transport equation for \( k \). Dissipation can be thought of as the rate at which the kinetic energy of the eddies is transferred to smaller scales and ultimately into heat through viscous action. The exact \( \varepsilon \)-equation can be derived from its exact definition:

\[
\varepsilon = -\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}
\]  

(2.13)

and a number of studies have used this expression to develop models (e.g. Mansour and Rodi, 1993; Howard, 1999; Jakirlić and Hanjalić, 2002). However, many of the terms in the exact \( \varepsilon \)-equation are difficult to model individually, and as such the model is generally seen as mostly empirical. Following earlier work by Chou (1945), Davidov (1961) and Harlow and Nakayama (1968), the most widely used variant has come from Launder and Sharma (1974), who proposed a refinement of the previous work by Jones and Launder (1972). It is robust and its behaviour is comparatively well understood. In the \( k - \varepsilon \) model the eddy viscosity is calculated as:

\[
\nu_t = C_\mu \frac{k^2}{\varepsilon}
\]  

(2.14)

where \( C_\mu \) is known as the constant of proportionality, defined empirically through consideration of flow under local equilibrium conditions.

The exact equation for the transport of turbulent kinetic energy may be calculated as one half the trace of the transport equation for the Reynolds stresses \( \overline{u_i u_j} \), which is itself derived directly from the Navier-Stokes equations. The turbulent kinetic energy equation
may be thus modelled as follows, where the substantive derivative is defined for a scalar \( \phi \) as

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + U_k \frac{\partial \phi}{\partial x_k},
\]

where \( \frac{Dk}{Dt} = d_k + P_k - \varepsilon \) (2.15)

where \( d_k \) is the diffusion of kinetic energy, which can be thought of as the transport of the fluctuations by viscosity and the fluctuations themselves. The pressure diffusion term, which appears in the exact equation for \( k \), is usually modelled together with viscous and turbulent diffusion using a gradient transport model as follows:

\[
d_k = \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_k u_i u_i - \frac{1}{\rho} u_k p \right) \sim \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \sigma_k \right) \frac{\partial k}{\partial x_j} \right] \quad (2.16)
\]

where \( \sigma_k \) is the effective Prandtl number, taken as \( \sigma_k = 1 \) for the Launder Sharma model.

The production of turbulent kinetic energy by the mean flow, \( P_k \), is the transfer rate of kinetic energy from the mean flow to the turbulence. This is represented exactly as:

\[
P_k = -u_i u_j \frac{\partial U_i}{\partial x_j} \quad (2.17)
\]

The corresponding (modelled) equation for the dissipation rate of turbulent kinetic energy is:

\[
\frac{D\varepsilon}{Dt} = d_\varepsilon + \varepsilon \frac{\varepsilon}{k} P_k - \varepsilon^2 \frac{\varepsilon^2}{k} \quad (2.18)
\]

with a similar expression for diffusion defined as:

\[
d_\varepsilon = \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] \quad (2.19)
\]

where the constants are: \( C_{\varepsilon 1} = 1.44 \), \( C_{\varepsilon 2} = 1.92 \), and \( \sigma_\varepsilon = 1.3 \). These constants were originally tuned for very simple turbulent flows. The term \( C_{\varepsilon 2} \) is calculated from decaying homogenous isotropic turbulence and \( C_{\varepsilon 1} \), which strongly influences the spreading rate of free-shear layers, is calibrated for a plane mixing layer. Several problems can arise from the \( \varepsilon \) equation since, even for a channel flow it does not correctly reproduce the dissipation, and several terms of its budget show discrepancies with a-priori DNS analysis (Manceau et al., 2000). Work was done by Gatski and Speziale (1997) to derive an anisotropic model for dissipation to correct this problem. Hanjalić and Launder (1980)
introduced a term to the dissipation equation with the purpose of differentiating the effects of irrotational and rotational straining on the energy transfer (production of $\varepsilon$).

The eddy viscosity proportionality constant, $C_\mu$, is defined by use of the local equilibrium assumption, which states that at high Reynolds number the rates of production and dissipation are of similar order of magnitude. This hypothesis was used by Bradshaw et al. (1967), together with the experimental observation that the stress-intensity ratio in the log-layer is of the order $-\bar{u}^2/k \approx 0.3$, to obtain the common value of $C_\mu = 0.09$.

When the Reynolds number of the flow is sufficiently large, (typically of the order of $Re \approx 10^5$) the eddy viscosity is several orders of magnitude greater than the molecular viscosity, allowing the influence of molecular viscosity to be neglected. At lower Reynolds numbers and, in particular, near to a wall, the eddy viscosity is of the same order as the molecular viscosity and, closer still to the wall, the flow is dominated by viscous effects. This occurrence leads to large gradients of velocity and turbulence profiles, and special treatments are often required at the wall to avoid the need to numerically resolve these, usually implemented in the form of a wall function. A modified form of the $k-\varepsilon$ model, referred to as a Low-Reynolds-number form, avoids the need to employ a wall function by including dependences on molecular viscosity, allowing the equations to be applied right the way up to the wall. However, this approach requires a very fine computational near wall resolution in order to capture correctly the large gradients in this region. In the popular $k-\varepsilon$ model of Launder and Sharma (1974), a modified form of the turbulent dissipation rate is proposed which reduces to zero at a solid boundary, and a damping function is applied to the eddy viscosity formulation to ensure the correct profile as the boundary is approached.

The advantages of both the $k-\varepsilon$ model and the $k-\omega$ model were combined in a hybrid $k-\varepsilon/k-\omega$ model known as the Baseline model (Menter, 1992, 1993). A low Reynolds number formulation of the $k-\omega$ model is used in the boundary layer and a version of the $k-\varepsilon$ model is used in the free shear layer. This is based on the observations that the $k-\varepsilon$ model is much less sensitive to the free-stream value of $\varepsilon$ than the $k-\omega$ model is to $\omega$. The Baseline model was constructed using a blending function, which appears as a coefficient in the equation. It is possible to derive an equivalent $\omega$ equation from the equations of $k$ and $\varepsilon$, since they are related by $\omega = \varepsilon/\beta^*k$ where $\beta^* = 0.09$. However, in the process of such a derivation, various cross derivative terms appear, and the inclusion or exclusion of this accounts for many of the important differences between the $\varepsilon$ and $\omega$
equations.

A further modification by Menter (1994) led to the Shear Stress Transport (SST) model. In addition to the above blending of $\varepsilon$ and $\omega$ equations there is now a limiter placed on the eddy viscosity to make the shear stress proportional to the kinetic energy in the boundary layer. The SST model has been demonstrated to perform well in a variety of flows, especially for boundary layer flows.

Non-Linear Eddy Viscosity Models

Inherent in all linear eddy viscosity models is the assumption that Reynolds stress anisotropy is directly linked to the mean strains as $a_{ij} = -(2\nu_t/k)S_{ij}$; implying that any predicted anisotropy vanishes as the mean strain goes to zero. Although this approximation is adequate for many simple flows, particularly those dominated by simple shear, this is violated in more complex flows, for example, flows with three-dimensional boundary layers; impinging flows, and flows with strong curvature.

Another limitation of the linear EVM lies in the calculation of the turbulent energy production rate, where the $P_k$ term can be shown to be proportional to the square of the strain rate. This is particularly problematic in the vicinity of stagnation points (a region where the strain invariant is very high), where levels of $k$ are seen to be massively over-predicted. This modelling error was initially reported by Craft et al. (1993) in their study of impinging jet flows, and a correction for this kind of flow was later proposed by (Durbin, 1996) in the form of a limiter on the turbulent time scale. The EVM formulation also restricts the production term to always take a positive value, another oversimplification that can be shown to be erroneous for certain flows.

An alternative approach is to express the anisotropy tensor $a_{ij}$ explicitly as non-linear polynomials of the mean velocity gradients. A more general expression for the anisotropy first suggested by Pope (1975) takes the form $a_{ij} = A_{ij}(S_{ij}, \Omega_{ij})$, where the vorticity tensor, $\Omega_{ij} = 1/2 \left( \partial U_i / \partial x_j - \partial U_j / \partial x_i \right)$, and $A_{ij}$ is a second order tensor. Pope used the Cayley-Hamilton theorem to show that if $a_{ij}$ is exclusively dependent upon $S_{ij}$ and $\Omega_{ij}$, it can be expressed in a maximum of ten tensorially independent groups. Although this expression contains terms up to fifth order, few non-linear eddy viscosity models include terms higher than third order, the complete list of which is as follows:
\[ a_{ij} = \beta_1 S_{ij} + \beta_2 (S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}) + \beta_3 (S_{ik}S_{kj} - \frac{1}{3}S_{ik}S_{kl}\delta_{ij}) + \\
\quad \beta_4 (\Omega_{ik}\Omega_{kj} - \frac{1}{3}\Omega_{ik}\Omega_{kl}\delta_{ij}) + \beta_5 (\Omega_{ik}S_{im}S_{mj} - S_{im}\Omega_{mj}) + \\
\quad \beta_6 (\Omega_{il}\Omega_{jm}S_{mj} - S_{il}\Omega_{lm}\Omega_{mj} - \frac{2}{3}S_{lm}\Omega_{mn}\Omega_{nj}\delta_{ij}) \quad (2.20) \]

where the coefficients \( \beta_i \) can be functions of \( k, \varepsilon \) and the independent invariants of \( S_{ij} \) and \( \Omega_{ij} \). For example, setting \( \beta_1 = -C_{\mu}\kappa/\varepsilon \) and \( \beta_{i>1} = 0 \) returns the linear stress-strain relationship. The remaining coefficients can be calibrated in two ways, the first of which is by reference to a number of fundamental flows. Speziale (1987) proposed a model calibrated for square ducts and pipe U-bends, while Nisizima and Yoshizawa (1987) developed another version for channel and Couette flow and Lumley et al. (1993) applied their NLEVM to rotating flows. All these models, amongst others proposed at around the same time, included terms up to second order only, and as they were calibrated for different flows, the constants varied widely.

It can be shown that the normal stress anisotropy in a plane channel flow is accounted for solely by the quadratic terms of equation 2.20, but Suga (1995) showed that cubic terms were necessary in order to predict correctly the Reynolds stress distributions due to streamline curvature or swirl. In response to this observation a cubic model was proposed by Craft et al. (1996b) which introduced quadratic and cubic functions of strain rate and vorticity. This model was tuned to several different reference flows: a homogenous shear flow, a fully developed swirling flow and a flow with streamline curvature. One model that does include fourth order terms is that of Lübcke et al. (2003), who found these additional terms to be influential in improving predicting of three-dimensional wall jets. However, in general there exists insufficient evidence to warrant the additional algebraic complexity of fourth and fifth order terms in NLEVMs and such models are rarely employed.

The other method of calibrating the constants \( \beta_i \) involves the inversion of the simplified transport equations for the Reynolds stresses themselves. By neglecting the transport effects, it is possible to obtain an explicit set of equations known as an explicit algebraic stress model. This approach is described in more detail later on.
2.1. Turbulence Modelling

2.1.3 Second Moment Closure

A major limitation of two equation models is a result of the six Reynolds stresses, $u_i u_j$, being represented by scalar values such as $k$ and $\varepsilon$, which brings important limitations on how accurately the stress anisotropy can be represented. The other fundamental problem is the assumption of an instantaneous equilibrium, implied by the EVM; a supposition implying that the Reynolds stresses can be calculated from local values of turbulence and velocity gradients. If a higher level of closure is provided, by calculating the transport of each Reynolds stress component individually, then it becomes possible to include non-local and history effects in the calculation. Launder et al. (1975) developed such a model which has become widely used and is sometimes referred to as the LRR model.

Reynolds Stress Models

Reynolds stress models require the calculation of the six independent stresses and an additional equation for the length scale (generally either $\varepsilon$ or $\omega$ is used). However, the improved physical content of this approach is offset by the increased computational cost of the solution of seven equations. The exact transport equation for the Reynolds stresses is as follows:

$$\frac{\partial u_i u_j}{\partial t} = P_{ij} + d_{ij} + \Pi_{ij} - \varepsilon_{ij} \quad (2.21)$$

where

$$P_{ij} = -u_i u_k \frac{\partial U_j}{\partial x_k} - u_j u_k \frac{\partial U_i}{\partial x_k}, \quad d_{ij} = \frac{\partial}{\partial x_k} \left( \nu \frac{\partial (u_i u_j)}{\partial x_k} - u_i u_j u_k \right),$$

$$\Pi_{ij} = -\frac{1}{\rho} \left( \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right), \quad \varepsilon_{ij} = 2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}. \quad (2.22)$$

where $P_{ij}$ is the production rate, $d_{ij}$ is diffusion, $\Pi_{ij}$ is the velocity-pressure gradient term and $\varepsilon_{ij}$ is the dissipation rate. The production and the molecular diffusion can be expressed exactly, but the remaining terms must be modelled. The velocity-pressure gradient correlation term, $\Pi_{ij}$ is commonly expressed as the sum of pressure-strain, $\phi_{ij}$, and pressure diffusion, $d_{ij}'$:

$$\Pi_{ij} = p \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) - \left( \frac{\partial u_j p}{\partial x_i} + \frac{\partial u_i p}{\partial x_j} \right) \quad (2.23)$$
In homogenous flows pressure-diffusion vanishes, and even in other cases it is generally only significant at the wall. It is thus common to assume this term to be negligible in high Reynolds number regions. Pressure-strain can be shown to have zero trace, thus it does not directly change the turbulent kinetic energy and does not appear in the transport equation for $k$. This term is responsible for the redistribution of the turbulent energy between the different stresses. It tends to redistribute the stresses towards an isotropic state and is thus often known as the ‘return to isotropy’ term.

A Poisson equation is used to relate the pressure to the velocity, and is then solved using the Green’s function for Laplace’s equation in unbounded space. This method introduces two-point correlations into the equation; a fact which substantiates the closure problems of RSMs because of their dependence on non-local effects. However, most of the models considered do not make much direct use of the resulting expression of $\phi_{ij}$, other than to identify two separate elements for modelling purposes.

The pressure-strain term is the focus of modelling in second moment closure, with a number of different approaches in existence. Hanjalić and Launder (1972) suggested a model for pressure-strain and the subsequent models of Launder et al. (1975) and Speziale et al. (1991) are all developments of this scheme. In the absence of body forces the pressure-strain term is usually modelled in two parts; one part involving only turbulence quantities, $\phi_{ij1}$, and another involving the mean strain rate, $\phi_{ij2}$:

$$\phi_{ij} = \phi_{ij1} + \phi_{ij2} \tag{2.24}$$

The first term, $\phi_{ij1}$, describes the interactions of turbulent eddies with other turbulent eddies and is independent of the strain rates, $S_{ij}$. It takes time for the energy changes due to rapid distortions in the flow to cascade down to the turbulent eddies before contributing to the transport equation, and as such this term is sometimes referred to as the ‘slow term’. The common form, proposed by Rotta (1951), is basically a linear relaxation of the anisotropy tensor, $a_{ij}$ to zero. For this reason it is also known as the return to isotropy term and takes the following form:

$$\phi_{ij1} = -C_1 \varepsilon a_{ij} \tag{2.25}$$

where $C_1$ commonly takes the value of 1.8, calibrated from experiments of free return to isotropy of initially strained turbulence. The second term, $\phi_{ij2}$, describes the interaction of the mean strain with turbulent eddies. It is known as the ‘rapid term’ since it responds
quickly to rapid distortions in the flow as it explicitly involves the deformation rate tensor $S_{ij}$. The flow is assumed to be quasi-homogenous, which holds fairly well for flows away from walls, although this hypothesis is strongly violated in regions close to the wall, where further modelling is required. Since it is the rates of strain which provide the main source for turbulent production, this term can be expected to directly modify the process of stress production, it has become common to approximate this term via the Isotropisation of Production (IP) model. It takes the following form:

$$
\phi_{ij2} = -C_2 \left( P_{ij} - \frac{2}{3} \delta_{ij} P_k \right)
$$

(2.26)

where $P_k = 1/2 P_{ii}$ and $C_2$ is calibrated from Rapid Distortion Theory. Other pressure-strain models have adopted more complex modelling approaches involving terms quadratic in $a_{ij}$ and $S_{ij}$. Li (1992) demonstrated that the linear assumption is invalid for flows with large degrees of curvature or strong anisotropy in his study of non-circular cross sectional ducts. As such, the quasi-non linear\(^2\) model of Speziale et al. (1991), and the Two Component Limit model of Craft et al. (1996a), which is cubic in $a_{ij}$, are able to provide a more accurate description of the redistribution. The Two Component Limit model has the additional benefit of not requiring any wall correction terms to account for wall effects.

Of the remaining terms in the exact Reynolds stress model, somewhat simpler modelling is generally applied. An isotropic assumption is commonly made for the dissipation as $\varepsilon_{ij} = 2/3 \varepsilon \delta_{ij}$ which is adequate for flows at high Reynolds number. However when the Reynolds number is lower or when approaching a wall, this is not exact. Diffusion, $d_{ij}$, can be modelled using the gradient diffusion hypothesis, proposed by Daly and Harlow (1970), which takes the following form:

$$
d_{ij} = \frac{\partial}{\partial x_k} \left[ (\nu \delta_{kl} + c_s u_k u_l) \frac{\partial u_i u_j}{\partial x_l} \right]
$$

(2.27)

An interesting modification to the standard RSM approach is the Hybrid RSM model of Basara and Jakirlić (2003), which addresses the convergence difficulties sometimes encountered as a result of the coupling between the Reynolds stress equations and the momentum equations. Their idea was to use an eddy viscosity model for the diffusion terms of the momentum equations whilst retaining the calculated Reynolds stresses in the sources terms of the $k$ and $\varepsilon$ equations. This approach has been shown to reduce the

---

\(^2\)so called quasi-linear since it features quadratic terms in $a_{ij}$ but not of $S_{ij}$
numerical stiffness and speed up convergence without significantly reducing the accuracy of results with respect to the full approach (see also Jakirlić, 2003).

**Algebraic Stress Models**

The cost and complexity of full second moment closures is a disadvantage from an industrial viewpoint, and in an attempt to reduce the associated computational cost Rodi (1976) proposed the ‘weak-equilibrium’ assumption in which the transport effects of stress anisotropy are assumed to be negligible. This is obtained from the transport equation for the Reynolds stresses, assuming isotropic diffusion as follows:

\[
\frac{Du_i u_j}{Dt} - d_{ij} \approx \frac{u_i u_j}{k} \left( \frac{Dk}{Dt} - d_k \right) = \frac{\bar{u}_i \bar{u}_j}{k} (P_k - \varepsilon) \tag{2.28}
\]

Substituting this approximation into the stress transport equations leads to a set of algebraic equations for the Reynolds stresses known as the Algebraic Stress Model:

\[
(P_k - \varepsilon) a_{ij} = P_{ij} - \frac{2}{3} \delta_{ij} P_k + \phi_{ij} \tag{2.29}
\]

These equations succeed in bringing improvements over EVM predictions and are less costly than the full RSM, but the neglect of the transport of stress anisotropy can adversely affect results for flows with rapid spatial variations where the turbulence is far from equilibrium conditions of \( P_k = \varepsilon \). Moreover, the implicit nature of the algebraic equations can lead to problems of numerical stiffness, which can offset the saving of CPU effort over solution of the full transport equations, and as such these models have not had widespread use.

More recently, improvements upon ASMs have been proposed that invert equation 2.29 so as to obtain explicit expressions for the stress anisotropies. This approach was introduced by Pope (1975) and is commonly known as an Explicit Algebraic Stress Model (EASM). Gatski and Speziale (1993) extended this method to three-dimensions and more recent examples include the model of Apsley and Leschziner (1998), Wallin and Johansson (2000) and Abe et al. (2002). These models are closely related to a full second moment closure, and in equilibrium conditions should give exactly the same solution as the RSM from which they were derived. However, as with ASMs, EASMs sometimes suffer the same disadvantages that result from the weak-equilibrium assumption.
2.1.4 Wall Effects

Wall Reflection Terms

The basic Reynolds stress model and its variants have been used with some success, showing considerable advantages over eddy viscosity models in flows in non-equilibrium (Launder, 1989). However, in wall-bounded flows the pressure-strain model requires an additional term, known as the wall reflection term, $\phi_{ijw}$, to account for damping of the wall-normal stress component. Models for this term have been proposed by Shir (1973) and Gibson and Launder (1978), but they depend on the distance to the wall and the wall normal direction, which pose problems for complex geometries.

Elliptic Relaxation

Durbin (1993) proposed an alternative approach to modelling the pressure-strain terms. Elliptic relaxation models aim to improve the representation of non-local effects such as wall-blocking and wall-echo, which are instantaneously felt far from the wall because the pressure adjusts through an elliptic equation (i.e., throughout the entire flow field). Elliptic equations for the pressure-strain and the pressure-diffusion terms can be derived from the fluctuating pressure equation that reproduce the non-local effects due to the presence of solid boundaries. In Durbin’s approach, therefore, in addition to $u_iu_j$ and $\varepsilon$, another set of variables, $f_{ij}$ is solved for which essentially represent the sum of the dissipation anisotropy $(\epsilon_{ij} - \frac{2}{3} \epsilon \delta_{ij})$ and redistribution terms in the RSM. However, the full $u_iu_j - f_{ij}$ model is seldom implemented because of the cost of incorporating the further six equations. A earlier study by Launder (1986) showed that it is the wall-normal stress component that provides the correct velocity scale near the wall, which led Durbin (1991, 1995) to develop a reduced form of the above the closure known as the $v^2 - f$ model. Only two extra equations are required compared to an EVM: one for the wall-normal stress $v^2$ and one for its corresponding elliptic redistribution, $f_{22}$. This model has been shown to give improved accuracy over EVMs for a range of wall bounded flows, although some numerical stiffness led to modifications from Lien and Durbin (1996) and Laurence et al. (2004).
2.2 Near-wall treatment

The presence of a solid wall affects a turbulent flow in a number of ways, and as such the treatment of wall boundary conditions and near-wall modelling requires particular attention. Viscous stresses are generally negligible in comparison to turbulent stresses far from a wall, but as the wall is approached the turbulent stresses are damped and the viscous stresses dominate.

There are two main approaches for accounting for these effects with turbulence models. Low-Reynolds-number models incorporate terms that are valid throughout the laminar and turbulent regions of the near wall flow. A fine near wall mesh is required to resolve the large gradients of turbulent flow properties \(^3\), which increases the computational requirements to solve the flow field, especially for flows with many walls.

The second approach involves a ‘high-Reynolds-number model’, which should be used with a coarse near wall mesh so that the near wall cell includes all of the viscous sublayer and part of the fully turbulent region of the boundary layer. Transport equations are solved in the main region of the flow and empirically based expressions known as Wall Functions are used in the wall bounding cells. It is common to base such wall functions on “universal” laws from simple flows such as a fully developed Couette flow (zero pressure gradient, shear driven flow). This enables the functions to be easily derived but also explains why they fail in more complex flows. In a zero pressure gradient boundary layer, for example, the near-wall mean \(U\)-velocity equation can be reduced to

\[
\frac{\mu}{\rho} \frac{\partial U}{\partial y} - \rho \overline{uu} = \tau_{wall}
\]

(2.30)

The viscous sublayer is the thin region immediately adjacent to the wall, \((y^+ < 5)\). Since the Reynolds stresses are zero at the wall, the wall shear stress is \(\tau_{wall} = \mu (\partial U / \partial y)\). In dimensionless terms the velocity, \(U^+ = U / u_\tau\), is equal to the wall distance \(y^+\) in this region.

In the fully turbulent region (\(y^+ \approx 30 \rightarrow y/\delta \approx 0.1\)) the viscous stresses are negligible and so equation 2.30 reduces to \(-\rho \overline{uu} = \tau_{wall}\). In order to obtain the velocity profile in this region, the log-law is formulated by combining the above equation with the mixing length hypothesis (equation 2.10) to obtain:

\(^3\)the first near wall node should be at around \(y^+ = 1\), and approximately 10 - 20 nodes should be placed below the log-layer. The non-dimensional wall distance is defined as \(y^+ = y u_\tau / \nu\), where \(u_\tau = \sqrt{\tau_{wall}/\rho}\)
where $C = 5.0$ and $\kappa = 0.41$. These near wall laws have been extensively verified using experimental and DNS data for simple flows although they have also been shown to fail when the flow moves away from the assumed local equilibrium conditions. All the standard wall functions make use of the above log-law for determining the wall shear stress $\tau_{wall}$, mostly using the velocity scale provided by $k^{1/2}$ to define $y^+$ and $U^+$. Early versions used $u_*$ as the velocity scale, but this was problematic in separated flow where $\tau_{wall}$, and hence $u_*$, are zero. The production and dissipation rates of $k$ change rapidly near the wall and so instead of taking the near-wall node value these are usually averaged over the near-wall cell. Many different approaches to this averaging have been proposed, giving rise to a wide range of wall-functions. The wall function proposed by Chieng and Launder (1980) introduced a linear variation of $k$ across the near-wall cell to calculate averaged $k$ generation and dissipation terms.

One of the main problems with the standard wall function approach is the high sensitivity of the results to the location of the first near wall node and the local grid density. The scalable wall function proposed by Grotjans and Menter (1998) is an industrial fix, aiming to remove this sensitivity to grid resolution. The basic idea behind the scalable wall function is to place a lower limit on the value of $y^+$ used in the log profile, i.e. $y^+ = \max(y^+, 11.06)$. It is therefore assumed that the wall coincides with the edge of the viscous sublayer, and so any nodes in the viscous sublayer are ignored. This allows wall functions to be applied on arbitrarily fine near-wall meshes and is particularly useful when the boundary layer detail is not the main objective.

A major weakness of log law type of wall functions is the assumed logarithmic velocity profile in the inner turbulent region. This assumes that the boundary layer is two-dimensional, in local equilibrium and in a zero-pressure gradient. Unfortunately, these assumptions are often invalidated; for example if the boundary layer is accelerated, or separated, or if there is an external force. An interesting alternative is the Analytical wall function proposed by Craft et al. (2002), which is able to avoid some of these issues, since it does not make use of the log law. Instead a simplified momentum equation is specified in the near-wall cell, which is then integrated, with a prescribed profile of $\nu_t$, to give an analytical profile for the velocity variation across the near-wall cell, from which quantities such as the wall shear stress can be evaluated.
2.3 Turbulence Modelling of Unsteady Flows

In order to meet the increasing demands from industry, turbulence models are often required to provide predictions of unsteady flows. The information gained from an unsteady calculation can offer a far deeper insight into the characteristics of a particular case. For example, an accurate description of fluctuating pressure signals can be used to provide an acoustic signal, or maximum instantaneous values of temperature can be important to predict fatigue, possibly indicating failure where the long-time averaged values would not.

A study into the shortfalls of linear eddy viscosity models in non-isotropic turbulence was undertaken by Rotta (1979) who examined the mis-alignment angle between the turbulent stress and mean strain rate tensors for three-dimensional thin shear layers. He observed that since the vector of the shear stress is in general not parallel to the vector of the mean velocity gradient, a more correct approach would be to adopt a three-dimensional version of the eddy viscosity relationship.

A more recent attempt to account for non-equilibrium effects is proposed by Olsen and Coakley (2001) who coupled a baseline two-equation model with an additional transport equation to account for lag effects in non-equilibrium steady state flows. The third equation prevents the eddy viscosity from responding instantaneously to changes in the mean strain rate field by relaxing its value gradually towards the equilibrium value, and some improvements over the baseline two-equation model are observed.

The steady state equilibrium assumptions at the foundation of the standard RANS models soon become inadequate for unsteady flows. The complex shifting of energy across the turbulent scales can result in a state of non-equilibrium, where energy production and dissipation are out of phase with each other. The energy spectra will be far from the equilibrium state defined by the characteristic intermediate range slope discovered by Kolmogorov (1941). Consequently, the modelling assumptions based upon the definition of an integral length scale and the cascade of energy become less valid.

Another method that attempts to address the issue of non-equilibrium is the Organised Eddy Simulation approach of Ha Minh and Kourta (1993). The local-equilibrium hypothesis is rejected and instead, it is proposed that the energy spectrum should be manually modified to preserve energy levels in unsteady flows. Vortex shedding in unsteady flows is typically characterised by high energy, low frequency modes of the spectrum, together
with a non-equilibrium inertial range, i.e. a slope of the energy spectrum that is not equal to $-5/3$. This behaviour can be approximated by manually reducing the value of the constant $C_\mu$ from 0.09 to 0.02, which effectively reduces the amount of energy in the scales associated with higher frequency turbulence. The constant $C_\mu$ was selected to be 0.02 by extraction of the diagonal anisotropy tensor from low Reynolds number DNS calculations and RSM calculations of a flow around an aerofoil (Braza, 2000; Hoarau et al., 2002).

Another study into aerofoil flows with very thin boundary layers led Spalart et al. (1997) to propose a hybrid RANS-LES model known as Detached Eddy Simulation (DES). The idea is to allocate the attached eddies in the boundary layer to a RANS model, whilst the detached regions, where the flow is separated, are simulated by LES calculations. The RANS turbulent length scale at each fluid element is calculated and compared to the spatial resolution of the mesh. When the RANS length scale is less that the grid resolution the calculation uses the RANS length scale, and when the RANS length scale is greater than the grid size LES is used to compute the flow. This approach requires careful modelling of the transition from the RANS to the LES treatment, which can lead to the spurious generation of eddy viscosity at the interface. Standard DES typically employs either the Spalart and Allmaras model or the Menter SST model, and therefore this approach will be subject to at least some of the same shortfalls experienced by the use of these models alone. A great deal of work is now focussing on the development of more stable methods of combining LES and RANS in order to reduce the cost of LES at high Reynolds numbers (see Davidson and Dahlstrom, 2005; Temmerman et al., 2005, for examples of this work). A promising idea from Menter and Egorov (2005) focuses on an oversight in the original length scale model of Rotta (1951) to formulate the Scale-Adaptive Simulation approach (SAS), which uses a length scale based on velocity gradients to sensitise the model to smaller scale turbulence. As such this is more like multi scale URANS model than a Hybrid approach, and does not involve the blending of RANS to LES, thus avoiding interface and grid sensitivity issues.

Typical RANS models make use of a single length scale, which is a significant oversimplification of the true nature of unsteady turbulent flows. Non-linearity in the inertial range implies that the turbulence scales are highly variable, and several multi scale turbulence models have thus been developed which solve a set of two or more different length scales in order to attempt to represent the different scales associated with different parts of the turbulence spectrum (see Schiestel, 1998, for a comprehensive guide to this work).
2.4 Selection of Unsteady Test Cases

It is the objective of this work to derive and test a turbulence model that is sensitive to the stress-strain lag inherent in an unsteady turbulent flow. A prudent selection of test cases is necessary in order to achieve this goal, since the number of cases which can be examined is constrained by both time and resources. It is envisaged that this work should provide the groundwork for a modelling approach that may feasibly be used in an industrial context, and as such the work will not be limited to purely academic cases. Moreover, due to the implicit simplicity of most academic cases, they can often, if necessary, be computed more precisely by using the more costly tools available to modellers, such as LES and DNS. Therefore a balance is sought here between simple test cases, and more complex industrial applications. The following paragraphs briefly describe the selection process of the test cases considered in this thesis, the literature for which is reviewed more completely in the Introduction of each results chapter.

The original idea for this work stems from a result of the investigation by Hadžić et al. (2001) on homogenous cyclic strains, in which the authors highlighted the ability of the Reynolds stress transport model to capture the stress-strain lag, in contrast to the standard $k – \varepsilon$ model which erroneously predicts that the stress and strain remain aligned at all times. This test case thus becomes the logical starting point in the validation process of the new model, since it returns results which are clearly different for the RSM and standard two equation models.

Subsequently, in preparation for future test cases which involve separation around a bluff body, it is necessary to apply the model to a wall bounded flow, so that a robust near wall treatment can be developed. The case of an oscillating channel flow has been studied extensively both experimentally and numerically, and this is an important example of flows which exhibit some stress-strain lag (see Scotti and Piomelli, 2001, for example). First the fully developed steady channel flow will be considered, before moving on to the oscillating channel flow itself, where separate LES calculations by Benhamadouche (2006) were performed specifically for this work, in order to obtain detailed streamwise profiles of the flow.

Although some stress-strain lag is apparent in the oscillating channel flows, it was expected that a more extreme example would be found in the separated flow behind a bluff body. Unfortunately, due to the highly unsteady nature of these flows, the available
experimental data is rather limited. Two cases were selected, which would appear to be of interest from an industrial perspective, and which clearly show the errors that arise as a consequence of turbulence modelling practices which make the assumption that the turbulent stress and mean strain rate tensors are directly aligned to one another. First, the case of the NACA0012 airfoil at an incidence angle of 20° (i.e. beyond stall) is considered, for which there exists some data from Favier et al. (1992) in the boundary layer of the flow. Secondly, the case of flow past a circular cylinder in a square duct was selected, which has been studied by Perrin et al. (2005a) and represents one of the more comprehensive sets of experimental data available for an unsteady turbulent flow. The cylinder case in particular will also enable a visual comparison to be made of the natural unsteadiness in the flow, in the form of the coherent turbulent structures in the wake of the cylinder, which are expected to be considerably over-damped as a direct consequence of the inappropriate modelling of the turbulence production provided by standard eddy viscosity models.
Chapter 3

Numerical Procedure

All the methods for resolving or modelling turbulent flow reviewed in Chapter 2 result in a set of nonlinear coupled partial differential equations for transport of momentum. Depending upon the exact closure method used, one or more additional transport equations must also be solved for the turbulent quantities. Non-linear partial differential equations are notoriously difficult to analyse, as very few analytical solutions exist, and so in CFD, numerical methods are required to approximate a solution. In the search for more accurate, more efficient solution algorithms, a plethora of approaches have been developed, and many choices must be made by the user, from the selection of the mesh, to the discretisation method, and the handling of gradients, convection, diffusion and boundary conditions. For a comprehensive review of the most widely used methods, see Ferziger and Perić (1999).

In this work the bulk of the calculations are performed using an unstructured finite volume code, Code_Saturne (Archambeau et al., 2004). The finite volume method is attractive for engineering applications since it is inherently conservative and can be used with unstructured or structured meshes.

3.1 The Finite Volume Method

The general transport equation for a scalar quantity $\phi$ in a compressible flow can be written as follows:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (U \phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi$$

(3.1)
where \( U \) is the transporting velocity, \( \Gamma \) is the diffusion coefficient and \( S_{\phi} \) stands for a source term, specific to the transported quantity. Terms \( a, b \) and \( c \) are the unsteady, convective and diffusive terms respectively. In Code_Saturne, the collocated arrangement for variables is used, where all variables are located at the centre of gravity of the cells (see Figure 3.1). The principle of the finite volume method is to subdivide the computational domain into a finite number of control volumes, \( \Omega_I \). The conservation equations are then integrated over each volume as follows:

\[
\frac{\partial}{\partial t} \left( \int_{\Omega_I} \phi \, d\Omega \right) + \int_{\Omega_I} \nabla \cdot (U\phi) \, d\Omega = \int_{\Omega_I} \nabla \cdot (\Gamma \nabla \phi) \, d\Omega + \int_{\Omega_I} S_{\phi} \, d\Omega \tag{3.2}
\]

Gauss’ divergence theorem is applied to the integrals of convection and diffusion, to transform the volume integral, \( \int_{\Omega} d\Omega \), into a surface integral, \( \int_{S} dS \), resulting in the following expression, where \( \mathbf{n} \) is the vector normal to the surface \( S \):

\[
\frac{\partial (\phi_I \Omega_I)}{\partial t} + \int_{S} \phi (U \cdot \mathbf{n}) \, dS = \int_{S} \Gamma (\nabla \phi) \cdot \mathbf{n} \, dS + S_{\phi} \Omega_I \tag{3.3}
\]

The unsteady term is approximated as \( \int_{\Omega} \phi \, d\Omega \approx \phi_I \Omega_I \), where \( \phi_I \) is the value of \( \phi \) at the centre of gravity of the element \( I \), and \( \Omega_I \) is the volume of the element. A similar approximation is made for the source term. Any source terms that can be written \( S_{\phi} \Phi_I \), where \( S_{\phi} \) is negative, are treated implicitly, thus enhancing numerical stability by augmenting diagonal dominance of the resulting solution matrix. The treatment of convection and diffusion is discussed below.

### 3.1.1 Handling of Convection and Diffusion Terms

The convection scheme concerns the discretisation of term \( b \) in equation 3.3. Stability of the convective terms is especially sensitive to the numerical scheme, and as such, a large number of methods have been proposed to avoid instabilities and to minimise numerical diffusion. The two most common convection schemes are the first order accurate upwind scheme (UDS), and the second order accurate centred difference scheme (CDS), both of which are available in Code_Saturne. Other more complex convection schemes exist which use higher orders of interpolation, e.g. QUICK (Leonard, 1979), UMIST (Leschziner and Lien, 1994); but they pose problems for the implementation into an un-
structured code \(^1\).

Since \textit{Code_Saturne} is an unstructured code, the convective and diffusive transport is calculated with the cell face as the frame of reference, unlike for structured codes which are able to use a Cartesian cell. Figure 3.2 shows a general configuration of an internal face between two adjacent elements \(\Omega_i\) and \(\Omega_j\). Point \(F\) is the centre of gravity of the face and points \(I\) and \(J\) are the centres of mass of the two adjacent elements. Points \(I'\) and \(J'\) are the projections of \(I\) and \(J\) onto the vector normal to the common face. Point \(O\) is the intersection of the vector \(IJ\) and the face. In order to compute the convection for a given element, the mass flux of the variable \(\phi\) is summed over each of the element faces:

\[
\int_S \phi (U \cdot n) \, dS \approx \sum_{\Omega_j \in \mathcal{N}(\Omega_i)} \phi_{ij} m_{ij} \tag{3.4}
\]

where \(\Omega_j \in \mathcal{N}(\Omega_i)\) represents the set of neighbouring elements, \(\Omega_j\), to the current element \(\Omega_i\). The mass flux, \(m_{ij}\), is the mass flux across the face between the elements \(\Omega_i\) and \(\Omega_j\), and is calculated as follows:

\[
m_{ij} = (U_{ij} \cdot n_{jj}) s_{jj} \tag{3.5}
\]

where \(s_{jj}\) is the area of the surface located between the elements \(\Omega_i\) and \(\Omega_j\), \(n_{jj}\) is the vector normal to this surface and \(U_{ij}\) is the value of \(U\) at the face. As can be seen from equation 3.4, it is also necessary to evaluate \(\phi_{ij}\), the value of \(\phi\) at the centre of the internal face between the two elements (point \(F\) on fig 3.2). Since \(\phi\) is only known at the cell centres, an approximation is required, which is obtained according to the selected discretisation scheme as detailed below.

**Upwind Differencing Scheme (UDS)**

For the first order UDS scheme (Courant et al., 1952), the value of \(\phi\) at a face is taken to be its value at the upwind neighbouring cell centre, i.e. either \(\phi_i\) or \(\phi_j\). This can be expressed as follows:

\[
\phi_{ij} = \frac{1}{2} \left[ (m_{ij} + |m_{ij}|) \phi_i + (m_{ij} - |m_{ij}|) \phi_j \right] \tag{3.6}
\]

\(^1\)A Second Order Linear Up-wind scheme (SOLU scheme) is also available in \textit{Code_Saturne} but not used in the present work.
which selects $\phi_{IJ} = \phi_i$ if $m_{IJ} > 0$ and $\phi_{IJ} = \phi_j$ if $m_{IJ} < 0$. This method is simple, stable and bounded, although it is numerically diffusive. Thus, unless the grid is very fine, an upwind scheme will introduce severe errors when the flow direction is oblique to the grid.

**Central Differencing Scheme (CDS)**

The second order CDS scheme considers a weighting of both $\phi_i$ and $\phi_j$, and improves the accuracy of approximation of $\phi_{IJ}$:

$$
\phi_{IJ} = \alpha_{IJ}\phi_i + (1 - \alpha_{IJ})\phi_j
$$  \hspace{1cm} (3.7)

where the weighting function is defined as $\alpha_{IJ} = J'F/I'J'$, which for a uniform grid will be one half.

**Flux reconstruction**

Both the UDS and the CDS schemes are implemented in *Code_Saturne* with some correction for grid non-orthogonality which may exist at an internal face, particularly if true unstructured grids are used (see Figure 3.2). Assuming that the gradient of $\phi$, denoted $\nabla \phi$, is known at the cell centre $I$, then the value of $\phi$ at point $I'$ is approximated as:

$$
\phi_{I'} = \phi_i + I'(\nabla \phi)_I
$$  \hspace{1cm} (3.8)

and thus, dependent upon the choice of convection scheme, equation 3.6 or 3.7 is modified so that $\phi_i = \phi_{I'}$ (and $\phi_j = \phi_{J'}$). To aid stability, $(\nabla \phi)_I$ can be replaced by the average of the gradients at points $I$ and $J$, i.e. $(\nabla \phi)_I + (\nabla \phi)_J)/2$

**Diffusion term**

The diffusion scheme concerns the discretisation of term $c$ in equation 3.3, which can be approximated as follows:

$$
\int_S \Gamma(\nabla \phi)_n dS \approx \sum_{\Omega_i \in N(\Omega)} \Gamma_{IJ}(\nabla \phi)_{IJ} \cdot n_{IJ} S_{IJ}
$$  \hspace{1cm} (3.9)

---

2The non-orthogonality is the angle between the normal to the face, $n_{IJ}$, and the vector which links the centres of two adjacent elements, $I'J'$. When these two vectors are co-linear, the face is said to be orthogonal, otherwise the face is non-orthogonal.
where \( \Gamma_{IJ} \) is the diffusion coefficient at the face centre, which is taken to be an average of the values stored at \( I \) and \( J \). The term \((\nabla \phi)_{IJ} n_{IJ}\), represents the normal gradient of \( \phi \) at the face centre \( F \), and is approximated as follows:

\[
(\nabla \phi)_{IJ} n_{IJ} = \frac{\phi_J - \phi_I}{I'J'}
\]  

(3.10)

3.1.2 Gradient Calculation

The calculation of the gradient of \( \phi \) at the cell centre, \( I \) of a computational cell is especially important in unstructured codes, which rely on accurate gradient approximations. Code_Saturne uses the gradient reconstruction method of (Musaferija and Gosman, 1996) to calculate the cell centre gradient \((\nabla \phi)_I\). Assuming that \( \phi_I \) is known, an expression for \((\nabla \phi)_I\) is given using Gauss’ theorem to be:

\[
\int_{\Omega} \nabla \phi \, d\Omega = \int_{S} \phi n \, dS
\]

(3.11)

this can be reformulated to express the gradient in terms of the values of \( \phi \) at the cell faces to give:

\[
\Omega_I (\nabla \phi)_I = \sum_{\Omega_{j\in\Omega_I}} \phi_{IJ} n_{IJ} S_{IJ}
\]

(3.12)

At this stage, the values at the faces, \( \phi_{IJ} \), are required to calculate the gradient, and vice-versa (in the flux reconstruction scheme), so equation 3.12 is solved in an iterative manner until the values of \((\nabla \phi)_I\) and \( \phi_{IJ} \) have converged. This is a costly method, and Code_Saturne offers an alternative method of estimating gradients using the Least Squares approach which, although significantly faster, is less robust. In this work the gradient reconstruction method is used for all calculations with Code_Saturne.

3.1.3 Boundary Conditions

Volume integrals are calculated in the same way for all internal elements, but boundary conditions are required at the limits of the computational domain in order to define the way in which fluxes are treated at a boundary. Figure 3.3 shows the general configuration of a boundary cell as treated in Code_Saturne. Boundary conditions fall into two main categories, outlined below.
Chapter 3. Numerical Procedure

Dirichlet boundary conditions

These conditions are typically used at an inlet or at a wall, where the value of the variable at the boundary face, \( \phi_F \), is assigned the value \( \phi_{\text{dir}} \); for example zero velocity at a wall. This takes the general form:

\[
\phi_F = \phi_{\text{dir}}
\]  
(3.13)

Neumann boundary conditions

Instead of fixing a value at the face itself, this condition implies that the normal gradient of the variable at the boundary face is fixed. This is used, for example, in outlet conditions and in faces on a symmetry plane, where a homogenous Neumann condition is set, \( (\nabla \phi)^{\text{neu}} = 0 \), which implies that the normal gradient to the boundary plane is zero. The general form in Code_Saturne is:

\[
\phi_F = (\nabla \phi)^{\text{neu}} F + \phi_F
\]  
(3.14)

Periodic boundary conditions

Periodic boundary conditions are treated differently since they are not actually treated as strict boundaries. Instead, a layer of ‘ghost cells’ are created on the opposite side of the boundary face, and so a periodic boundary face is considered as an internal face. It then becomes straightforward to impose a translation to the corresponding periodic faces.

3.1.4 Solving the Equations

Finally the fully discretised form of equation 3.1, to be solved for each element, can be written as:

\[
\Omega_i D_i \phi_i + \sum_{\Omega_j \in \Omega_i} \left[ \phi_{ij} m_{ij} - \Gamma_{ij} \left( \frac{\phi'_j - \phi'_i}{J} \right) \right] S_{ij} = \Omega_j S'_i
\]  
(3.15)

where \( D_i \) includes the time derivative and any implicit source terms, and \( S'_i \) includes the explicit source terms \( S \). An iterative algorithm is then used to solve equation 3.15. Regardless of the convection scheme adopted an extra up-wind term is added to both sides of the equation, to allow for the use of a Jacobi matrix inversion algorithm, which has the desirable stability properties of an up-wind matrix. This leads to the linear system
\[ A \mathbf{x} = B, \] where \( B \) is the right hand side of equation 3.15 and \( \mathbf{x} \) is the vector of the unknowns. The matrix \( A \) is not symmetric but is conditioned by an upwind scheme.
Chapter 4

Turbulence Models

4.1 Introduction

This chapter describes in more detail the exact forms of the turbulence models employed in the comparisons reported in this thesis. The bulk of the results in the thesis involve a three-way comparison, between the standard SST model of Menter (1994), the new SST-$C_{as}$ model (to be described in Chapter 5) and the quasi-linear Reynolds stress transport model of Speziale et al. (1991) (SSG). These three models are employed in the 3D unstructured finite volume solver, Code Saturne.

For the work on simple homogenous strains in Chapter 6, a simple 1D time marching solver was used and some additional models were implemented. A high-Reynolds-number forms of the linear $k-\varepsilon$ model and the cubic NLEVM of Craft et al. (1996b) were used. Two Reynolds stress models were also implemented, namely the standard model of Launder et al. (1975) (LRR) and the quasi-linear SSG model. Since these calculations were made for homogenous flows with trivial boundary conditions, some simplifications were possible in the implementation of these models.

4.2 Eddy Viscosity Models

Work in Chapter 6 employs the high Reynolds number $k-\varepsilon$ model of Launder and Spalding (1974) and the cubic NLEVM proposed by Craft et al. (1996b), whilst the remaining chapters use the SST model of Menter (1994) for comparisons with the new SST-$C_{as}$ model.
4.2.1 The High-Reynolds-Number $k - \varepsilon$ Model

The form of the standard $k - \varepsilon$ model used is shown previously in equations 2.15 and 2.18, with the eddy viscosity defined as in equation 2.14.

4.2.2 The Cubic NLEVM

The cubic eddy viscosity model of Craft et al. (1996b) is used in the work on simple homogenous strains in Chapter 6. It employs the same $k$ and $\varepsilon$ transport equations as described for the $k - \varepsilon$ model above, but uses a non-linear stress-strain relation as defined below, with model coefficients given in Table 4.1:

\[
a_{ij} = -\frac{\nu_t}{k} S_{ij} + c_1 \frac{\nu_t}{\varepsilon} \left( S_{ik} S_{kj} - \frac{1}{3} S_{kl} S_{ki} \delta_{ij} \right) + c_2 \frac{\nu_t}{\varepsilon} \left( \Omega_{ik} \Omega_{j} - \frac{1}{3} \Omega_{ik} \Omega_{k} \delta_{ij} \right) + c_3 \frac{\nu_t \varepsilon}{\varepsilon^2} \left( S_{kl} \Omega_{ij} + S_{lj} \Omega_{ki} \right) S_{kl} + c_4 \frac{\nu_t \varepsilon}{\varepsilon^2} \left( \Omega_{ij} S_{lm} S_{mj} + S_{ij} \Omega_{lm} \Omega_{mj} - \frac{2}{3} S_{lm} \Omega_{mn} \Omega_{nl} \delta_{ij} \right) + c_5 \frac{\nu_t \varepsilon}{\varepsilon^2} \left( S_{ij} S_{kl} \Omega_{kl} \right) + c_6 \frac{\nu_t \varepsilon}{\varepsilon^2} \left( S_{ij} \Omega_{kl} \Omega_{kl} \right), \tag{4.1}
\]

where the strain rate, vorticity rate, strain rate parameter and turbulent viscosity are defined as:

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right), \tag{4.2}
\]

\[
S = \frac{k}{\varepsilon} \sqrt{S_{ij} S_{ij}/2}, \quad \Omega = \frac{k}{\varepsilon} \sqrt{\Omega_{ij} \Omega_{ij}/2},
\]

\[
\eta = \max(S, \Omega), \quad \nu_t = C_\mu \frac{k^2}{\varepsilon},
\]

with variable $C_\mu$ defined as:

\[
C_\mu = \frac{0.3 \left[ 1 - \exp(-0.36 \exp(0.75 \eta)) \right]}{1 + 0.35 \eta^{1.5}}, \tag{4.3}
\]

\[
\begin{array}{cccccccc}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
-0.2 & 0.4 & 1.04 & -40C_\mu^2 & 0 & -40C_\mu^2 & -40C_\mu^2
\end{array}
\]

Table 4.1: Coefficients of the cubic NLEVM of Craft et al. (1996b)
4.3 Reynolds Stress Models

4.2.3 The Shear Stress Transport Model (SST)

The Shear Stress Transport model of Menter (1994) switches between the \(k-\omega\) model near the wall and the \(k-\varepsilon\) far from it. It introduces a blending function to obtain a smooth transition between the two. The two transport equations that are solved for turbulent kinetic energy, \(k\), and turbulent frequency, \(\omega\), are as follows:

\[
\frac{Dk}{Dt} = \frac{\partial}{\partial x_j} \left[ \frac{u_i u_j}{\nu} \frac{\partial U_i}{\partial x_j} \right] - \beta^* k \omega + \frac{\partial}{\partial x_j} \left[ (\nu + \nu_t) \left( \frac{\partial k}{\partial x_j} \right) \right] \tag{4.5}
\]

\[
\frac{D\omega}{Dt} = \frac{\gamma}{\nu_t} \frac{\partial}{\partial x_j} \left[ \frac{u_i u_j}{\nu_t} \frac{\partial U_i}{\partial x_j} \right] - \beta \omega^2 + \frac{\partial}{\partial x_j} \left[ (\nu + \nu_t) \left( \frac{\partial \omega}{\partial x_j} \right) \right] + 2 (1 - F_1) \sigma_{\omega^2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} \tag{4.6}
\]

where the function \(F_1\) is defined as:

\[
F_1 = \tanh \left( \arg_1 \right), \tag{4.6}
\]

\[
\arg_1 = \min \left[ \max \left( \frac{\sqrt{k}}{0.09 \omega y}; \frac{500 \nu}{y^2 \omega} \right), \frac{4 k \sigma_{\omega^2}}{C_{D_{k\omega}} y^2} \right],
\]

\[
C_{D_{k\omega}} = \max \left( 2 \sigma_{\omega^2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}; 10^{-20} \right)
\]

and the turbulent eddy viscosity is defined as follows:

\[
\nu_t = k \min \left( \frac{1}{\omega} \frac{a_1}{\|S\| F_2} \right), \tag{4.7}
\]

\[
F_2 = \tanh (\arg_2^2),
\]

\[
\arg_2 = \max \left( 2 \sqrt{k} \frac{1}{0.09 \omega y}; \frac{500 \nu}{y^2 \omega} \right),
\]

where \(y\) is the distance to the nearest wall and the model coefficients are shown in Table 4.2. The wall boundary condition for \(\omega\) proposed by Menter is given by:

\[
\omega_w = 10 \frac{6 \nu}{\beta_1 y^2} \tag{4.8}
\]

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(\sigma_{k1})</th>
<th>(\sigma_{\omega1})</th>
<th>(\beta_1)</th>
<th>(a_1)</th>
<th>(\beta^*)</th>
<th>(\kappa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\beta_1}{\beta^*} \frac{\sigma_{\omega1} e^2}{\sigma_{k1}})</td>
<td>0.85</td>
<td>0.50</td>
<td>0.0750</td>
<td>0.31</td>
<td>0.09</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Table 4.2: Coefficients of the SST model of Menter (1994)

4.3 Reynolds Stress Models

The standard Reynolds stress model of Launder et al. (1975) and the quasi-linear model of Speziale et al. (1991) are used in the work on simple homogenous strains in Chapter
6. Although both models are available in *Code_Saturne* only the SSG model was used in subsequent chapters. The model implementation in *Code_Saturne* includes the Simple Gradient Diffusion (SGD) model of Shir (1973) instead of the Generalized Gradient Diffusion (GGD) model of Daly and Harlow (1970), and the complete model is as follows:

\[
\frac{Du_i u_j}{Dt} = \frac{\partial}{\partial x_k} \left[ \left( \nu + C_s \frac{k^2}{\epsilon} \right) \frac{\partial u_i u_j}{\partial x_k} \right] - \frac{u_i u_k}{\partial x_k} \frac{\partial U_j}{\partial x_k} - \frac{u_j u_k}{\partial x_k} \frac{\partial U_i}{\partial x_k} + \Pi_{ij} - \frac{2}{3} \epsilon \delta_{ij}
\]

\[
\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_k} \left[ \left( \nu \delta_{kl} + C_\epsilon \frac{k}{\epsilon} \frac{\partial u_i u_j}{\partial x_k} \right) \frac{\partial \epsilon}{\partial x_k} \right] - C_{\epsilon 1} \frac{\epsilon}{k} \frac{\partial U_k}{\partial x_j} - C_{\epsilon 2} \frac{\epsilon^2}{k}
\]

(4.9)

where the pressure-strain term \(\Pi_{ij}\) is modelled as follows, with the coefficients defined in Table 4.3:

\[
\Pi_{ij} = - \left( C_1 \epsilon + C_1^* P_k \right) a_{ij} + C_2 \epsilon \left( a_{ik} a_{kj} - \frac{1}{3} \delta_{ij} A_2 \right) + \left( C_3 - C_3^* \sqrt{A_2} \right) k S_{ij}
\]

\[
+ C_4 k \left( a_{ik} S_{jk} + a_{jk} S_{ik} - \frac{2}{3} a_{lm} S_{lm} \delta_{ij} \right) + C_5 k \left( a_{ik} \Omega_{jk} + a_{jk} \Omega_{ik} \right)
\]

(4.10)

<table>
<thead>
<tr>
<th></th>
<th>(C_1)</th>
<th>(C_1^*)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_3^*)</th>
<th>(C_4)</th>
<th>(C_5)</th>
<th>(C_s)</th>
<th>(C_\epsilon)</th>
<th>(C_{\epsilon 1})</th>
<th>(C_{\epsilon 2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSG</td>
<td>1.7</td>
<td>0.90</td>
<td>1.05</td>
<td>0.8</td>
<td>0.65</td>
<td>0.625</td>
<td>0.2</td>
<td>0.22</td>
<td>0.18</td>
<td>1.44</td>
<td>1.92</td>
</tr>
<tr>
<td>LRR</td>
<td>1.8</td>
<td>0</td>
<td>0</td>
<td>0.8</td>
<td>0</td>
<td>0.873</td>
<td>-0.655</td>
<td>0.22</td>
<td>0.18</td>
<td>1.44</td>
<td>1.83</td>
</tr>
</tbody>
</table>

Table 4.3: Coefficients of the SSG and LRR pressure-strain models

### 4.3.1 Near Wall Treatment

The Scalable Wall Function of Grotjans and Menter (1998) is available in *Code_Saturne* for use with the Reynolds stress models described above. This approach limits the value of \(y^+\) at the near wall cell centre to a minimum of 11.06, thereby effectively shifting the intersection between the viscous sublayer law and the log law to coincide with the position of the first node.

The process of implementing the wall function involves first calculating the velocity scale \(u_k\) at the near wall node, before calculating \(y^+\) which is used in the log-law to calculate \(u_t\), leading to the wall shear stress \(\tau_w\). This algorithm is shown below, where
\( y_{lim}^+ = 11.06, \kappa = 0.41 \) and \( C = 5 \).

\[
\begin{align*}
  u_k &= (C_\mu)^{\frac{1}{2}} \sqrt{k} \\
  y_b^+ &= \max\left(\frac{u_k y_b}{\nu}; y_{lim}^+\right) \\
  u_r &= \frac{1}{\kappa} \log(y_b^+) + C \\
  \tau_{wall} &= \rho u_r u_k 
\end{align*}
\]

(4.11)

The physical interpretation of the minimum limit imposed on \( y_b^+ \) procedure is to shift the wall to coincide with the limit of the logarithmic layer.

The flux at the wall boundary face is then computed by altering the velocity at the boundary face itself \( U_F \):

\[
\begin{align*}
  \left. \frac{dU}{dy} \right|_{wall} &= \frac{\tau_{wall}}{\mu} \\
  U_F &= U_I - IF \left( \frac{\tau_{wall}}{\mu} \right) 
\end{align*}
\]

(4.12)
Chapter 5

The $C_{as}$ Model for Stress-Strain Lag

5.1 Introduction

In contrast to eddy viscosity models, the Reynolds stress model calculates an exact production which is explicitly linear in the mean strain rate. Indeed, one can write exactly $P_k = C_{as} k \|S\|$ where $C_{as}$ is a non-dimensional parameter representing the degree of alignment between stresses and strains:

$$C_{as} = -\frac{a_{ij} S_{ij}}{\|S\|}$$  \hspace{1cm} (5.1)

The aim of the present work is to develop a model for the stress-strain lag, $C_{as}$, which appears to be a key parameter in rapidly evolving flows.

Both the stress anisotropy, $a_{ij}$, and strain rate, $S_{ij}$, are $3 \times 3$ symmetrical tensors, and the associated eigenvectors are therefore real and orthogonal. The anisotropy tensor has zero trace and is dimensionless by definition, whereas the strain rate tensor is an inverse time scale and has zero trace only in the condition of incompressibility, which is assumed for this work. As previously stated, an EVM assumes that these two tensors are aligned. However, DNS data from channel flow and from homogeneous shear flows can be used to see that even in these simple cases, this assumption is not true.

The alignment of two $2 \times 2$ symmetric tensors can be measured by a single angle between the sets of orthogonal principle axes, thus implying that the alignment for all quasi-2D flows is representable by a single dimensionless scalar. For a $3 \times 3$ tensor, the orientation is defined by 3 angles\(^1\), which can be obtained through calculation of the

---

\(^1\)Commonly referred to as the set of Euler angles
eigenvectors. The orientation of each tensor allows the mis-alignment to be given as the difference between each Euler angle, and so three scalar values are necessary to define the full stress-strain mis-alignment in a fully 3D flow. Analysis of the tensorial alignment between the strain rate tensor \( S_{ij} \) and the turbulent stress anisotropy tensor \(-a_{ij}\) has been used extensively to gain an insight into complex energy transfer mechanisms\(^2\), and also in the development of subgrid-scale stress models in Large Eddy Simulation (see Tao et al., 2000; Horiuti, 2001; Bergstrom and Wang, 2005).

The calculation of the eigenvalues of both strain and stress anisotropy tensors, per node, per timestep, is too expensive for large calculations. A simpler approach is to consider the inner product of the two original tensors, to give a scalar measure of the alignment rather than an angle, as:

\[
\cos(\kappa_{as}) = \frac{a_{ij}S_{ij}}{\|a\|\|S\|} \tag{5.2}
\]

where \(\kappa_{as}\) is the scalar measure of alignment, and the invariants \(\|S\|\) and \(\|a\|\) are defined as:

\[
\|a\| = \sqrt{a_{ij}a_{ij}} \quad \text{and} \quad \|S\| = \sqrt{2S_{ij}S_{ij}}, \tag{5.3}
\]

This simplification was used by Bergstrom and Wang (2003) in an \(a\)-priori analysis of subgrid-scale models, and was reported to provide a useful approximation. Peak values of \(\cos(\kappa_{as}) \pm \sqrt{1/2}\) will occur when \(a_{ij}\) is aligned with \(S_{ij}\). The definition of \(C_{as}\) from equation 5.1 is therefore bounded by \(\pm \|a\| / \sqrt{2}\).

The strategy adopted within the present work is to develop a transport equation that could be solved to obtain values for the parameter \(C_{as}\). The resulting values could be used in the evaluation of \(P_k\), within a EVM framework, in order to capture some of the features of stress-strain mis-alignment, but at a much smaller computational cost than employing a full stress transport model.

### 5.2 Model Derivation

The derivation of the proposed transport equation for \(C_{as}\) is shown here in detail. From the definition of \(C_{as}\) in equation 5.1 its total derivative can be obtained using the product
5.2. Model Derivation

The substantive derivative for $a_{ij}$ in equation 5.4 can be obtained from transport equations for the Reynolds stresses $\overline{u_iu_j}$ and the turbulence kinetic energy $k$, as follows:

$$\frac{Da_{ij}}{Dt} = \frac{1}{k} \left( \frac{D\overline{u_iu_j}}{Dt} - \frac{\overline{u_iu_j}Dk}{k} \right)$$  \hspace{1cm} (5.5)

$$= \frac{1}{k} \left( P_{ij} + \Pi_{ij} - \varepsilon_{ij} - \left(a_{ij} + \frac{2}{3} \delta_{ij}\right)(P_k - \varepsilon) \right) + \text{diffusion terms}$$

Where the diffusion terms are those arising from the difusive terms in the stress and $k$ equations. If the tensorial dissipation is approximated with an isotropic model as $\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij}$, then rearranging equation 5.6 leads to

$$k \frac{Da_{ij}}{Dt} = \left( P_{ij} - \frac{2}{3} P_k \delta_{ij} \right) - a_{ij} (P_k - \varepsilon) + \Pi_{ij} + \text{diffusion terms}$$  \hspace{1cm} (5.6)

which can be inserted into equation 5.4 to give

$$\frac{DC_{as}}{Dt} = -\frac{S_{ij}}{k ||S||} \left[ \left( P_{ij} - \frac{2}{3} P_k \delta_{ij} \right) - a_{ij} (P_k - \varepsilon) + \Pi_{ij} \right]$$

$$- \frac{1}{||S||} \frac{DS_{ij}}{Dt} \left( a_{ij} + \frac{2S_{ij}C_{as}}{||S||} \right) + \text{Diff}_{C_{as}}$$  \hspace{1cm} (5.7)

Although the term Diff$_{C_{as}}$ is not formally a purely difusive term for the quantity $C_{as}$, in the present approach these terms involving diffusion of $\overline{u_iu_j}$ and $k$ have been combined and can be approximated using a simple gradient diffusion model, where $\sigma_{cas} = 0.2$:

$$\text{Diff}_{C_{as}} = \frac{\partial}{\partial x_k} \left[ (v + \sigma_{cas} \nu_t) \frac{\partial C_{as}}{\partial x_k} \right]$$  \hspace{1cm} (5.8)

The production term, $P_{ij}$, can be written in terms of $a_{ij}, S_{ij}$ and $\Omega_{ij}$ as

$$P_{ij} = -k \left( \frac{4}{3} S_{ij} + a_{ik} S_{kj} + S_{ik} a_{kj} + a_{ik} \Omega_{jk} + \Omega_{ik} a_{jk} \right)$$  \hspace{1cm} (5.9)
which, when combined with a linear or quasi-linear pressure-strain model, $\Pi_{ij}$, leads to
the form of the transport equation for $C_{as}$:

$$
\frac{DC_{as}}{Dt} = \frac{\varepsilon}{k} C_{as} + \alpha_1 \bar{\varepsilon} ||S|| C_{as}^2 - \frac{S_{ij}a_{ik}a_{kj}}{\eta} \\
+ \left( \alpha_3 + \alpha_3^* \sqrt{A_2} \right) ||S|| + \alpha_4 S_{ij}a_{ik}S_{jk} \\
+ \alpha_5 \left( ||S|| \right) - \frac{1}{||S||} \frac{DS_{ij}}{Dt} \left( a_{ij} + \frac{2S_{ij}C_{as}}{||S||} \right) \\
+ \frac{\partial}{\partial x_k} \left[ \nu + \sigma_{cas} \nu \right] \frac{\partial C_{as}}{\partial x_k}
$$

(5.10)

The source terms in equation 5.10 are numbered for future reference and the constants $\alpha_1$, 
$\ldots$, $\alpha_5$ are related to those in the original underlying pressure-strain model via:

$$
\alpha_1 = (1 + C_1) \quad \alpha_1^* = (1 - C_1^*); \quad \alpha_2 = C_2 \quad \alpha_3 = \frac{\left( \frac{4}{3} - C_3 \right)}{2} \\
\alpha_3^* = \frac{C_3^*}{2} \quad \alpha_4 = 2 (1 - C_4) \quad \alpha_5 = 2 (1 - C_5)
$$

(5.11)

Table 4.3 lists the values of the model constants from the LRR and SSG pressure-strain
models ($C_1, \ldots, C_5$) and the corresponding constants calculated for the $C_{as}$ model ($\alpha_1$, 
$\ldots$, $\alpha_5$) are given in Table 5.1.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_1^*$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_3^*$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{as}$ (LRR)</td>
<td>-0.80</td>
<td>0</td>
<td>0</td>
<td>0.267</td>
<td>0</td>
<td>0.254</td>
<td>0.69</td>
</tr>
<tr>
<td>$C_{as}$ (SSG)</td>
<td>-0.70</td>
<td>-1.90</td>
<td>1.05</td>
<td>0.267</td>
<td>0.325</td>
<td>0.75</td>
<td>1.60</td>
</tr>
</tbody>
</table>

Table 5.1: Coefficients of the $C_{as}$ model, based on the SSG and LRR pressure-strain models
5.3 Model Implementation

The transport equation for $C_{as}$ shown in equation 5.10 was fully implemented into Code_Saturne. The substantive derivatives of the rate of strains were calculated explicitly as:

$$\frac{DS_{ij}}{Dt} = \frac{S^n_{ij} - S^{n-1}_{ij}}{\Delta t^n} + U_k \frac{\partial S^n_{ij}}{\partial x_k} \quad (5.12)$$

where $U_k \frac{\partial S^n_{ij}}{\partial x_k} \equiv \nabla \cdot (U S^n_{ij})$ is discretised as described in section 3.1.1, in the chapter on Numerics. Note that equation 5.10 is not in closed form as a model for $a_{ij}$ is still required, not only for pressure-strain model related terms, but also for the ‘exact’ stress anisotropy variation in term 7. In this work, values of the turbulent stress anisotropy $a_{ij}$ appearing in the $C_{as}$ equation are provided by one of two different methods:

- $C_{as}$-EVM: the anisotropy is of the form, $a_{ij} = -2S_{ij} \frac{kC_{ij}}{\varepsilon}$ (i.e. from a linear EVM formulation)
- $C_{as}$-NLEVM: the model of Craft et al. (1996b) was selected, where equations 4.1, 4.2 and 4.3 were adopted to approximate the anisotropy. Note that these values were only used in the source terms of the $C_{as}$ transport equation, and not in the momentum equations.

5.3.1 Near Wall Treatment

The $C_{as}$ equation as described above requires special treatment in the near-wall region as a consequence of the modelling of the pressure-strain and other terms, which are used in the derivation of the $C_{as}$ model (see Section 5.2). No viscous damping terms were included at the derivation stage and as such, the underlying stress model can be expected to be inaccurate up to a distance of $y^+ = 80 \sim 100$ away from the wall, as discussed by Durbin and Pettersson-Reif (2001). In this work two approaches were developed for the near wall treatment of $C_{as}$ which are described here.

**Fixed value in log-layer with $C_{as} = 0.3$ at the wall.**

In order to prevent possible errors in the near wall region from adversely affecting the bulk unsteady motion, it was decided in this first approach to apply a constant value of $C_{as}$ in

\footnote{The strain rate tensor $S_{ij}$ should not be confused with the notation for the area of a face between two cells $S_{IJ}$.}
the log-layer, effectively switching off the $C_{as}$ model in this region. A value of $C_{as} = 0.3$ was selected, corresponding to the value one would expect to find in an equilibrium shear layer, where the strain rate parameter, $\eta = k/\varepsilon \cdot ||S|| = 3.33$: which occurs in the log-layer when the production to dissipation ratio, $P_k/\varepsilon = 1$, as follows:

$$P_k = kC_{as} ||S|| \equiv C_{mu} \frac{k^2}{\varepsilon} ||S||^2$$ (5.13)

$$C_{as} \equiv C_{mu} \frac{k}{\varepsilon} ||S|| = C_{mu} \eta$$

$$C_{as} \equiv 0.3$$

A Dirichlet boundary condition of $C_{as} = 0.3$ was thus applied, and this value was also imposed at nodes up to a distance of $y^+ = 80$ from the wall. This is the simplest of the two approaches tested, but restricts the $C_{as}$ model from affecting the near wall region.

Many standard two equation low-Reynolds-number models reduce $C_{mu}$ near to the wall. The $C_{as}$ model has the potential to naturally reproduce this near-wall damping if an appropriate low-Reynolds number RSM, such as the Two Component Limit model of Craft et al. (1996a), is selected as the underlying stress model. However, this would be expected to require significant tuning of the model coefficients. The decision to restrict the effect of the $C_{as}$ model in the near-wall region has been taken to allow the benefits of the stress-strain lag modelling to be clearly established, prior to fine tuning for low Reynolds number effects.

**Near wall damping with $C_{as} = 0$ at the wall.**

Preventing the model from becoming active near the wall can be a reasonable trade-off for flows in which the bulk unsteady turbulent motion is away from the wall, as in flow past bluff bodies, for example (see Chapters 9 and 10). However, this approach will limit the beneficial input from the $C_{as}$ model in cases where the production of turbulence is concentrated in the near-wall region. In the work on oscillating channel flows in Chapter 8, it will be shown that an accurate prediction of the turbulence in the unsteady near-wall flow region is crucial to the correct modelling of the whole flow. In an attempt to compensate for the lack of wall reflection terms and other wall damping in the underlying stress transport model, a damping function, $f_{cas}$, was proposed of the form:

$$f_{cas} = \tanh(e^{y^+ - 80})$$ (5.14)
which acts on the entire RHS of equation 5.10, and is shown in Figure 5.1 along with the Van Driest damping function for comparison. It provides a much stronger near wall damping than the Van Driest function, influencing the flow up to a distance of around $y^+ = 80$ from the wall, at which point the problems associated with a high Reynolds number pressure-strain model are negligible. The damping function was calibrated with respect to fully developed channel flows at three different Reynolds numbers, as well as the oscillating channel flows analysed in Chapter 8.

### 5.3.2 Numerical Treatment

During the course of the development and implementation of the $C_{as}$ model, various limiters were applied to ensure numerical stability of the model.

**Timescale limiter**

From equation 5.10, it can be seen that term 7 will pose numerical problems if the strain rate invariant, $\|S\|$, becomes zero when the rest of the term is non-zero. To avoid this problem $\|S\|$ was limited. The limiter on the turbulent timescale proposed by Durbin (1991) was used to limit both the turbulent timescale $\tau = k/\varepsilon$ and the mean flow timescale $1/\|S\|$ as follows:

$$\tau_{lim} = \max(\tau; 6 \sqrt{\frac{y}{\varepsilon}}) \quad (5.15)$$

$$\|S\| = \max(\|S\|; \frac{\vartheta}{\tau_{lim}})$$

where $\vartheta = 10^{-6}$, which effectively also limits the strain rate parameter $\eta$ to a minimum of $\vartheta$.

**Reynolds stress limiter**

In order to ensure that the Reynolds stress approximations from both the linear and cubic models were realizable, the standard weak and strong conditions of realizibility were imposed. The former requires non-negative energies and the latter requires that the Schwarz inequality, which states that the cross-correlation of two components is bounded by the magnitude of autocorrelations is satisfied (see Schumann, 1977, for a review of this topic).
These were implemented in the following form, which was applied to the stresses going into the source terms of the \( C_{as} \) equation only:

\[
\begin{align*}
\overline{u_i^2} & \geq 0 \\
\overline{u_3^2} & = 2k - \overline{u_1^2} - \overline{u_2^2} \\
\overline{u_\alpha u_\beta} & \leq \sqrt{\overline{u_\alpha^2} \cdot \overline{u_\beta^2}} \quad \text{for } \alpha \neq \beta
\end{align*}
\]

**Limiter on \( C_{as} \)**

From the definition of \( C_{as} \) in equation 5.1, it becomes apparent that this quantity is bounded by \( \pm \sqrt{a_{ij}^2/2} \) and, by considering the limits of \( ai_j^2 \) for an axisymmetric expansion flow, \( \pm \sqrt{4/3} \):

\[
\frac{a_{ij}a_{ij}}{2} = \frac{1}{2} \left( \frac{\overline{u^2 + v^2 + w^2}}{k} \right)^2 - \frac{4}{3} = \frac{4}{3} \quad (5.17)
\]

A stricter bound emerges from the realizability condition of Durbin (1996) who applied a limit to \( \nu_t \), within a linear EVM framework, to ensure that all normal stresses should be non-negative:

\[
0 \leq -2\nu_t \lambda_{max}^s + 2/3k 
\]

where \( \lambda_{max}^s \) is the maximum eigenvalue of \( S_{ij} \), which can be shown to satisfy the condition \( \lambda_{max}^s < \sqrt{||S||^2 / 3} \) for incompressible flows \(^4\). Substituting this limit into equation 5.18 yields the condition \( \nu_t \leq k / \sqrt{3} ||S|| \). Interpreting this bound on \( \nu_t \) as one on \( C_{as} \), now leads to:

\[
-\frac{1}{\sqrt{3}} \leq C_{as} \leq \frac{1}{\sqrt{3}} \quad (5.19)
\]

**5.4 The \( k - \varepsilon - C_{as} \) Model**

The introduction of \( C_{as} \) into the standard form of the \( k - \varepsilon \) model described in Section 4.2.1 requires only a single modification to the transport equations for turbulent kinetic

\(^4\)This appears by considering an axisymmetric expansion flow, and calculating \( ||S|| \) from the diagonalise matrix: which gives \( 2 \left( (\lambda^s/2)^2 + (\lambda^s/2)^2 + (-\lambda^s)^2 \right) = ||S||^2 \).
5.5 The SST-\(C_{as}\) Model

energy (equation 2.15) and dissipation (equation 2.18) where the turbulent viscosity, \(\nu_t\) is redefined as:

\[
\nu_t = \frac{C_{as}k}{||S||}
\]  

(5.20)

The rate of strain invariant appearing in the denominator comes from the re-definition of the production term as \(P_k = kC_{as}||S||\). In this work the \(k - \varepsilon - C_{as}\) model is used only in Chapter 6 for the 1D cases of homogenous flows, where convection and diffusion are zero. A simple time marching code was employed to compute the flows in this chapter, where inner iterations were used to ensure the terms of the \(C_{as}\) equation were fully converged before progressing to the subsequent timestep.

The simple nature of these flows meant that this model was sufficiently accurate although, as commented earlier, it is recognised that the \(k - \varepsilon - C_{as}\) model would not be adequate for more complex flows. It should thus be regarded as a development model, and was never fully implemented into Code_Saturne. Instead, the more complex testcases subsequently studied were computed using the SST-\(C_{as}\) version of the model.

5.5 The SST-\(C_{as}\) Model

The fully implemented SST model described in section 4.2.3 requires only small modifications to incorporate the \(C_{as}\) model. Initially, the modification was intended to be applied to the production rate of turbulence kinetic energy term only, but it can be applied in a more coherent manner by means of a simple modification to the turbulent eddy viscosity in equation 4.7, as follows:

\[
\nu_t = k \min \left( \frac{1}{\omega}; \frac{a_1}{||S||F_2}; \frac{C_{as}}{||S||} \right)
\]  

(5.21)

The value of \(C_{as}\) in equation 5.21 is limited to \(\pm 0.31\) for the calculation of the production terms in equation 4.5, while when evaluating diffusion terms, the absolute value, \(|C_{as}|\), is used to avoid negative values which could lead to numerical difficulties. When the near wall values of \(C_{as}\) are fixed as described in section 5.3.1, a Dirichlet boundary condition of \(C_{as} = 0.31\) is used instead of 0.30 in order to be consistent with the near wall behaviour of equation 5.21, where the constant \(a_1 = 0.31\) and the blending function, \(F_2\), is equal to unity in the wall region.
Solution strategy

In the results in this thesis, the standard SST model was used alone for the first few timesteps in order to initialise the calculation for the SST-$C_{as}$ model. No numerical difficulties were encountered with the modelling framework presented in this chapter, and a smaller timestep was not required. A full central differencing scheme was employed in the discretisation of the momentum equations, as is commonly used in LES and DES.

In general the additional computational expense, per timestep, of the SST-$C_{as}$ model over the SST model was of the order of $10 - 15\%$. 
Chapter 6

Homogenous Flows

6.1 Introduction

In this chapter, the $C_{as}$ model is applied to constant and then time dependent homogenous flows. In order to provide a simple validation of the newly derived model, the response of the $C_{as}$ model to a constant shear flow is calculated analytically. Subsequently, the initial response to a homogenous shear is then investigated and compared to results from existing models.

In order to demonstrate the benefits of the $C_{as}$ modelling scheme for unsteady flows, time dependent strained flows are then examined. A cyclic homogenous shear flow is first examined, and then the case of cyclic strain, which is a flow of direct relevance to the prediction of turbulent flow inside a piston engine.

The calculations in this section were carried out using a simple 1D code, where the equations were solved in a time-marching fashion without the need for spatial discretisation, because of the flow homogeneity.

6.2 Constant Homogenous Shear

Nearly all turbulence models have been tuned to return good predictions for homogenous shear flows, since they are common components of many practical flows. Experimental data for such flows can be obtained by placing a series of flat plates parallel to the flow direction, which increase linearly in length across the direction normal to the flow. This has the effect of creating a flow where the only non zero strain is $s = 2S_{12} = \partial U_1 / \partial x_2$.

For this case the $C_{as}$ model in equation 5.10 reduces to the following simple expression,
where \( \tau = k/\varepsilon, A = a_{12}, B = a_{11} \) and \( C = a_{22} \):

\[
\frac{\alpha_1 C_{as}}{\tau} + \alpha_1^2 C_{as}^2 s = \frac{\alpha_2}{\tau} A(B + C) - \left( \alpha_3 + \alpha_3^* \sqrt{2(A^2 + B^2 + C^2 + BC)} \right) s
\]

\[
-\frac{\alpha_4}{4} (B + C) s - \frac{\alpha_5}{4} (C - B) s
\]

(6.1)

Tavoularis and Karnik (1989) made measurements of a homogenous shear flow with a production to dissipation ratio \( P_R = 1.6 \pm 0.02 \). The corresponding strain-rate parameter \( \eta = s\tau = P_R/(-a_{12}) = 5 \) for this flow, and an average of their data for the stress anisotropy is shown as follows:

\[
a_{ij} = \begin{bmatrix}
0.36 \pm 0.08 & -0.32 \pm 0.02 & 0 \\
-0.32 \pm 0.02 & -0.22 \pm 0.05 & 0 \\
0 & 0 & -0.14 \pm 0.06
\end{bmatrix}
\]

(6.2)

Equation 6.1 can be rearranged into a quadratic form for \( C_{as} \), which is solved using the average values for \( a_{ij} \) given in equation 6.2. With the coefficients as detailed in Table 5.1 for the \( C_{as} \) (SSG) model, this returns a value of \( C_{as} = 0.340 \), close to the expected value of \( C_{as} = -a_{12} = 0.32 \); within the stated range of values. The slight overprediction is not entirely surprising, since the model constants of the SSG model were calibrated from an earlier set of data for homogenous turbulent shear from Tavoularis and Corrsin (1981), which when used instead of the values in equation 6.2, return the slightly more accurate value of \( C_{as} = 0.314 \).

Many experimental studies, and more recently DNS calculations, have been performed on simple shear flows and as such there is a range of data available at different rates of shear. One test of a turbulence model is thus its ability to predict the stress anisotropy resulting from a given applied strain. Suga (1995) was able to compute explicit expressions for the stress anisotropy predicted by his full cubic non-linear eddy viscosity model (equation 4.1) as follows:

\[
a_{11} = \left( \frac{c_1}{12} + \frac{c_2}{2} + \frac{c_3}{12} \right) C_\mu \eta^2
\]

\[
a_{22} = \left( \frac{c_1}{12} - \frac{c_2}{2} + \frac{c_3}{12} \right) C_\mu \eta^2
\]

\[
a_{12} = -C_\mu \eta^2
\]

(6.3)

The predictions, using his proposed model coefficients \( c_1, c_2, c_3 \) and \( C_\mu \) are plotted against various experimental results in Figure 6.1.
where \( \eta = \sigma \tau \) and \( C_\mu \) is calculated from strain sensitive expression in equation 6.3. In his work, Suga compared several other NLEVMs and was able to show that the cubic model performed much better in this case. The final state of stress anisotropy predicted by a model of this type at any given shear, \( \eta \), is reached when the transported turbulent quantities have reached an asymptotic ‘equilibrium’ value, i.e. equilibrium in the sense that their transport equations are balanced.

### 6.2.1 Initial Response to Homogenous Strain

When considering the development of a homogenous shear flow, however, a set of explicit stress equations will not respond in the correct manner to an applied strain, since the stress transport effects are grouped together and approximated in terms of the transport of turbulent kinetic energy, \( k \). In a full Reynolds Stress transport model the stresses are transported individually, and as such they are able to respond at different rates to the strain. They thus incorporate both transport and component history effects, which explicit expressions are unable to do. For this reason, in the present study, it is informative to calculate the response of a model to a developing shear flow over time.

Figure 6.2 shows the evolution of the strain rate parameter, \( \eta \), for different turbulence models, in response to three different initial values; \( \eta_0 = 2.6, 6 \) and 10. Results from a DNS calculation by Sarkar (1995) are included for the case \( \eta_0 = 2.6 \). The linear \( k - \epsilon \) model shows the most rapid evolution in each case, reaching its equilibrium state the soonest. It also has the lowest asymptotic value, lying considerably beneath the level the DNS appear to approach, implying that the turbulent timescale, \( \tau \), is lower than it should be, corresponding to a higher level of turbulence. This is an indication that the \( k - \epsilon \) model overpredicts the turbulence under constant strain, and can be verified from consideration of the transport equation for \( \eta \), valid for both EVMs and RSMs:

\[
\frac{d \eta}{d(St)} = -(C_{\epsilon 1} - 1)(-a_{ij}S_{ij})\eta + (C_{\epsilon 2} - 1) \quad (6.4)
\]

For the linear EVM, the term \(-a_{ij}S_{ij}\), which comes from the production term, becomes \( C_\mu \eta \), allowing the asymptotic limit to be obtained as:

\[
\eta_\infty = \left. kS \right|_{\infty} = \sqrt{\frac{(C_{\epsilon 2} - 1)}{C_\mu(C_{\epsilon 1} - 1)}} = 4.82, \quad (6.5)
\]
which corresponds to the value shown in figure 6.2. It is this feature that can lead to over-
production of turbulent kinetic energy, in regions of stagnation for example, and several
proposals have been made to limit the production, which have the effect of raising the
asymptotic limit to a larger value (see Menter, 1994; Durbin, 1995; Guimet and Laurence,
2002, as examples).

In comparison, the cubic $k – \varepsilon$ model takes a longer time to equilibrate, and the curve
for $\eta_0 = 2.6$ corresponds very well to the DNS data. This is perhaps not surprising since
the model was calibrated for such flows. However, compared to the results from the two
Reynolds stress models, it can be seen that the return to the fixed value is slower, and
the value itself is somewhat higher. The two RSM models exhibit a behaviour almost
identical to one another, despite their differences. The $k – \varepsilon – C_{as}$ model recovers in much
the same time as the RSMs, although it too reaches a lower asymptotic value\(^1\). This could
be corrected with a limit as described for the linear $k – \varepsilon$ model.

It is important to note that in the case of larger initial strain rates, $\eta_0 = 6, 10$, the $C_{as}$
model response closely matches the RSM behaviour over the time $St = 0 \rightarrow 10$, whereas
the linear and cubic models are unable to do so alone. A closer examination of the stress
evolution in this period can offer some explanation for this improvement.

Figure 6.3 displays the evolution of the stress anisotropy for $\eta_0 = 2.6$ and 10. It can be
seen that both the linear and cubic $k – \varepsilon$ models begin at non-zero values, which they are
obliged to do since they will return the “equilibrium” value of anisotropy corresponding
to any given value of $\eta_0$. They do not subsequently evolve a great deal, reaching limiting
values of $a_{ij}$ that correspond to those shown in Figure 6.1 for the specified $\eta$. The only
evolution in these models arises from transport effects in the $k$ and $\varepsilon$ equations, whereas
the RSM incorporates a separate evolution for $a_{12}$ as:

$$\frac{da_{12}}{d(St)} = -\frac{4}{3} - \frac{C_1 a_{12}}{\eta} + C_3 + (a_{11} + a_{22})(C_4 - 1) + (a_{22} - a_{11})(C_5 - 1)$$  \hspace{1cm} (6.6)

With zero normal stress anisotropy, the above equation can be integrated, and the initial
behaviour of $a_{12}$ close to $St = 0$ can thus be approximated as:

$$a_{12} = \left(\frac{4}{3} - C_3\right) \frac{\eta_0}{C_1} \left(e^{-\frac{C_1}{\eta_0} St} - 1\right)$$  \hspace{1cm} (6.7)

\(^1\)The $k – \varepsilon – C_{as}$ model uses the same values for $C_{e1}$ and $C_{e2}$ as the standard $k – \varepsilon$ model. The discrepancy
seen between the values of $\eta_\infty$ for these models is a result of the slight overprediction of $C_{as}$ for homogenous
shear described in Section 6.2
which is zero at $St = 0$ and holds until around $St = 1$. This is significant since, unlike the RSM, the EVM expression $a_{12} = C_\mu \eta$ is non-zero at $St = 0$. As expected, the RSMs both deliver a reasonable prediction for the development of $a_{ij}$ when compared to the DNS data. The $C_{as}$ model is also able to capture the correct behaviour of $a_{12}$ since it is also zero at $St = 0$.

Other common tests on turbulence models involve the response to constant plane strains, axisymmetric contraction and axisymmetric expansion. The majority of Reynolds stress models are able to provide a reasonable prediction of such an evolution for relatively low strains. At larger values of strain, the turbulence behaves in a different manner, and other techniques such as Rapid Distortion Theory (RDT) are required to account for the mechanisms in play.\(^2\)

In its present form, the $C_{as}$ model does not aim to provide the correct stress anisotropy, since it employs the linear stress-strain relation (although, as will be seen later, it may be beneficial to use an explicit stress approximation in order to provide the transport equation of $C_{as}$ with the correct balance of terms), but the model’s benefit is offered in the form of the response of the turbulence energy generation rate to the spatial and temporal variation of strain rates. For a simple shear flow, the value of $C_{as}$ should be equal to the value $-a_{12}$, and as such, the transport equation for $C_{as}$ can be thought of as a transport equation for $-a_{12}$ in this case. Traditional eddy viscosity type of stress approximation models are inherently unable to account for the history effects of turbulence at a given point if they are derived as functions of local values only.

### 6.3 Time Dependent Homogenous Strains

The response of turbulence to time-dependent strains has long been studied in turbulence theory and modelling. In the sections below, first a simple cyclic shear will be considered, before examination of the industrially relevant case of a cyclic strain, as is dominant in the cylinders of internal combustion engines, which will be considered in Section 6.3.2.

The idea at the origin of the development of the $C_{as}$ model stems from an analytical study of the dramatic differences between the behaviour of an eddy viscosity model and a Reynolds stress transport model in cyclic homogeneous strain (Hadžić et al., 2001).\(^2\)

---

\(^2\)Rapid Distortion Theory is not addressed in this work. See Hunt and Carruthers (1990) and Cambon and Scott (1999) for recent reviews of RDT, which was introduced by Batchelor and Proudman (1954)
Indeed, the EVM leads to the erroneous indefinite increase of turbulence while the RSM predicts that the turbulence disappears. In this section this test case is revisited and it is found that the use of the $C_{as}$ equation alters the behaviour of the EVM so that it returns results that are shown to be in close agreement with those from the RSM.

### 6.3.1 Homogenous Cyclic Shear

A cyclic shear of the following form $s = dU/dy$ is applied, where

$$s = 10 \cos(\Omega t) \quad (6.8)$$

and $\Omega = 2\pi/T$, with the cyclic period $T$ here chosen to be $T = 1$ for convenience. Solutions were examined for the initial conditions $\tau_0^* = 10$ and $\tau_0^* = 1$, where $\tau^* = \tau/T$ is the nondimensional turbulent time scale. Figure 6.4 displays the evolution of $\tau^*$ and the ratio of production to dissipation for the different models being tested. For the case $\tau_0^* = 10$, the turbulent timescale is predicted by all models to fall from its initial value at around the same rate. However, crucially, both the linear $k-\varepsilon$ model and the cubic $k-\varepsilon$ models reach a non-zero asymptotic value after around $t/T = 2$. This is in contrast to the behaviour of both the RSM and the linear $k-\varepsilon-C_{as}$ models, which continue to decrease until $\tau^* = 0$.

For the case $\tau_0^* = 1$, the different responses are even more evident. The linear $k-\varepsilon$ models predict that $\tau^*$ grows away from $\tau_0^*$, while the RSM and the $k-\varepsilon-C_{as}$ model predict that it decays from $\tau_0^*$. The cubic $k-\varepsilon$ predicts that $\tau^*$ does decrease a small amount, but at a much lower rate than that of the RSM. This behaviour can be clarified with reference to the evolution of $P_k/\varepsilon$, which shows that both original eddy viscosity models have a positive production while the RSM and $C_{as}$ models return values that oscillate around zero. This contrasting behaviour is due to the implicit inability of the eddy-viscosity model framework to allow a negative production.

The linear EVM gives a formulation for the production term as $P_k = -\bar{u}_i\bar{u}_j S_{ij} = 2\nu_t S_{ij} S_{ij}$, which must return a positive value on account of the quadratic product of $S_{ij}$ and the positive bound of $\nu_t$.

The cubic model gives a more complex expression for production of the form:

$$P_k = S_{ij} \left( 2/3k \delta_{ij} + a\nu_t S_{ij} + b\nu_t f(\tau, S_{ik} S_{kj}) + c\nu_t f(\tau^2, S_{ij} S_{kl} S_{kl}) \right), \quad (6.9)$$
6.3. Time Dependent Homogenous Strains

where $a$, $b$ and $c$ are constants and $\tau$ is the turbulent timescale. The leading term of the cubic model production term is positively bounded due to the same reasoning as in the linear case. However, in this case, the terms cubic in $S_{ij}$ can, in principle, become negative. The influence of the leading term can be reduced by the introduction of a strain dependent functional form of $C_\mu$, for example, but can only be reduced to a small positive number, and since $C_\mu$ will also appear in the higher order terms it remains difficult for the production term to become negative. This can explain why the results for the cubic model in Figure 6.4 show that the turbulent timescale falls below that of the linear model, eventually reaching a lower positive mean value.

The Phase Lag of Stress and Strain

In such cyclic flows, the transport of turbulence is influential in that it ensures the turbulent stresses are out of phase with the mean flow itself. The turbulent stresses are constantly in a non-equilibrium state of redistribution, reaching peak values that lag those of the mean flow field.

Figure 6.5 offers a simple representation of the different production terms, indicating how, when various phase lags exist between the mean strain and the turbulent stress anisotropy, the production will become negative over at least part of the phase. In fact, for the present case of simple shear, this lag is at a maximum of $\pi/2$, which implies that there will be zero net production over a full cycle. This is the reason why the Reynolds stress model predicts that the turbulence, and therefore the turbulent timescale, will continue to decrease until it is destroyed. As can be seen, the $C_{as}$ scheme is equally able to capture correctly the zero net production and the eventual destruction of the turbulence, at what appears to be at around the same rate as the full RSM.

Figure 6.6 plots the evolution of turbulent kinetic energy for the case $\tau_0^* = 1$ on a log-log plot. Here, the different responses are clearly visible; both eddy viscosity models grow exponentially, ultimately reaching the constant rate of growth indicated by the asymptotic value of the turbulent timescale in Figure 6.4, while the Reynolds stress model and the $C_{as}$ model both predict a decay of $k$. The $C_{as}$ model predicts that the turbulence disappears at a greater rate than that of the Reynolds stress model, a difference most likely due to the lack of stress anisotropy in the case of the $k - \varepsilon - C_{as}$ framework. Indeed, it was verified that the $k - \varepsilon - C_{as}$ model produces a decay curve very similar to the RSM when the RSM equations are used only to provide the stresses in the $C_{as}$ transport equation.
The phase lag between stress and strain can be seen from Figure 6.7, where the top plot shows the variation of the strain rate, $s$ and the derivative of the strain rate over one period of the cycle. The middle plot compares the prediction of the stress anisotropy $a_{12}$ for all models, where it can be seen that the linear and cubic $k - \varepsilon$ models predict a variation directly in phase with the strain (negative since $a_{12} \propto -s$). Although the cubic model adjusts to predict peak values of $a_{12}$ that are more in line with those from the RSM, it is unable to respond correctly to the change of sign of the strain and consequently provides an erroneous profile for the majority of the phase. The $k - \varepsilon - C_{as}$ profile is in good agreement with the RSM profile despite a slight overestimation in the region where $s = 0$. The bottom plot shows the periodic behaviour of $C_{as}$ itself, which gives a profile similar to $a_{12}$ but with the sign reversed every half phase, resulting from the expression $s/|s|$ contained in the definition for $C_{as}$. The small overprediction of $C_{as}$ is also evident in the bottom plot, which disappears when using the RSM stresses in an a-priori fashion in the $C_{as}$ equation, as can be seen, which serves as a validation of the derivation of the $C_{as}$ model.

The budget of the $C_{as}$ equation is examined for this flow in Figure 6.8 for both the RSM, where the terms in the equation are calculated a-priori, and the $k - \varepsilon - C_{as}$ model (the numbering of the terms is with respect to equation 5.10). It can be seen that there is a good agreement between the two models for terms 1,5 and 6, while the sum of terms 3 and 4 is not captured well by the $C_{as}$ model, since normal stress anisotropy is zero with this scheme (term 2 has been omitted from the plot since it is close to zero over the whole period). The failure of the linear $k - \varepsilon - C_{as}$ model to capture correctly the sum of terms 3 and 4 brings into question the need to incorporate a more accurate explicit algebraic set of stress approximations into the transport equation for $C_{as}$, which will be addressed later on. It is also noted that the overprediction of $C_{as}$ seen in figure 6.7 is mainly due to this issue.

The significance of term 7, which contains the term $ds/dt$ can be also be seen from Figure 6.8, where it is shown to be zero over most of the period except for brief intervals corresponding to $t/T \sim 0.25$ and 0.75, where it takes a large negative value. This negative value is associated with the abrupt change of sign of $C_{as}$ seen in Figure 6.7 for the same phase locations, and is due to the strain rate falling to zero ($|s| = 0$) while the expression $-a_{12} ds/dt$ is negative and non-zero. This term may thus pose numerical problems, since its
analytical value tends towards negative infinity \(^3\), although a careful timestep selection avoided any such problems in the present work. Arguably, an infinite value is unlikely to be reached in a more realistic case involving convection and diffusion of the mean flow since \(\|S\|\) is rarely exactly equal to zero, and a lower bound could be applied to \(\|S\|\) in this expression as described in Section 5.3.2 to avoid any numerical problems. It should also be noted that when this singularity occurs, it has no consequences on velocities or kinetic energy, since \(C_{as}\) will be multiplied by a zero strain.

6.3.2 Homogenous Cyclic Strain

For the case of a cyclic variation of the imposed strain, the stress field has been shown to exhibit a hysteresis (Hadžić et al., 2001). Such a flow is dominant in the cylinders of internal combustion engines. The real flow in such applications is unsteady and highly turbulent with complex structures which change as the engine undergoes its cycle and, due to strain-sensitive phenomena such as vortex breakdown, adequate turbulence modelling is imperative in order to predict accurately the combustion process. In the work by Hadžić et al. (2001), the authors demonstrate that although much current industrial CFD uses commercial codes employing standard \(k - \varepsilon\) models, the EVM provides a direct link between stress and strains which leads to erroneous results due to inadequate modelling of the generation of turbulence by irrotational strain.

In the work by Hadžić et al. (2001), the authors highlight work by Cambon and Scott (1999), who investigated Rapid Distortion Theory analysis of initially isotropic turbulence subjected to plane strain in which the strain direction was subsequently reversed. Cambon and Scott showed that the principal axes of the stress tensor do not coincide with the local strain tensor, but rather the accumulated strain tensor, \(I_{ij} = \int S_{ij} dt\). The present work has already confirmed the unsuitability of the EVM for cyclic shear flows, and the ability of the \(C_{as}\) model to correct this. In view of the observations by Cambon and Scott it is interesting to note that the substitution, \(P_k = kC_{as} \|S\|\), appearing in the \(k\) transport equation of the \(k - \varepsilon - C_{as}\) model, will also yield the accumulated strain tensor \(I_{ij}\) upon integration in time.

\(^3\)From equation 6.8, the time derivative of \(s\) is \(s' = -10\pi \sin(2\pi t)\), and therefore \(s'/|s|\) becomes \(\pi \tan(2\pi t)\); infinite for \(t = n\pi/2\) where \(n\) is an odd integer.
Flow Description

The evolution of turbulent kinetic energy in a general case of homogeneous turbulence subjected to a cyclic compression and expansion is considered here. The control volume can be thought of as being in the centre of a very large compression chamber as depicted in Figure 6.9, thus excluding secondary motions and wall effects. Although this is far from a real case, this idealised flow allows for simple numerical solutions of the problem, illustrating the importance of the treatment of the production mechanism of turbulence.

We consider a typical internal combustion engine case with a piston stroke of $1 - 1/r = 4/5$, where $r$ is the compression ratio. The imposed strains are $S_{11} = 2/3s$ and $S_{22} = S_{33} = -1/3s$ in order to ensure continuity. The time dependent strain, $s = dU/dx$ is taken as:

$$s = \frac{\Omega \cos(\Omega t)}{\sin(\Omega t) + \frac{5}{4}},$$  

(6.10)

and the derivative of the strain is:

$$\frac{ds(t)}{dt} = \Omega^2 \left[\left(\frac{\sin(\Omega t)}{\sin(\Omega t) + \frac{5}{4}}\right) + \left(\frac{\cos(\Omega t)}{\sin(\Omega t) + \frac{5}{4}}\right)^2\right]$$  

(6.11)

In this case $U$ is the local axial mean flow velocity, $x$ denotes piston direction and $t$ time from zero. The external parameter is the piston motion period $T \equiv 2\pi/\Omega$, which is used to non-dimensionalise $t$. The variation of both $s$ and $ds/dt$ are shown in Fig. 6.10.

Numerical solutions were obtained for an EVM, a NLEVM, an RSM and the $C_{as}$ model for the same case of homogenous cyclic compression. For a demonstration of the differences between these models, it is not crucially important which particular EVM or RSM is used. Figure 6.11 thus shows results from the standard $k - \varepsilon$, the cubic $k - \varepsilon$ of Craft et al., the SSG RSM and the $k - \varepsilon - C_{as}$ model.

It can be seen from Figure 6.11: row 1, that when the $k - \varepsilon$ model is employed the turbulent kinetic energy increases rapidly, since the production is always positive. This occurs since the stress and strain are directly in phase, with their maxima occurring at the same phase locations. The results for the cubic $k - \varepsilon$ model (row 2) follow a similar trend. Conversely, when the RSM is used, (row 3) the kinetic energy decays, due to a zero net production averaged over each cycle, which, combined with the dissipation, leads to an overall net decay. These results also indicate a phase lag of $\pi/2$ between the maxima of stress and strain, which was not captured by either of the eddy viscosity
models. In this case there is a stark difference between the results from the two different turbulent closure schemes, brought about by the failure of the EVM and the NLEVM to decouple the turbulent stresses from the strains. Under homogeneous conditions the contribution of both convection and diffusion is zero, thus this difference is due in large to the representation of the production term. As already discussed in Section 6.3.1, a negative production, or a back transfer of energy from turbulent motion to the mean flow, can not be achieved in a two equation context, whereas the independent calculation of turbulent stresses as in an RSM does not prevent this.

When the $k - \varepsilon - C_{as}$ model is used (Figure 6.11: row 4) the results of the RSM are well reproduced, despite some differences in the decay rate of the turbulence. There is clearly a phase lag present between the stress $\overline{u_1^2}$ and the strain $S_{11}$, although the stress curves do show a kink near the start of the calculation. The $C_{as}$ model’s performance in this case is largely due to the $ds/dt$ term in the $C_{as}$ equation, similarly noted for the case of cyclic shear in the previous section. Indeed, it is the change of sign of $C_{as}$ which allows the production to become negative and therefore permit a zero net contribution over one period to the production terms of the turbulence transport equations. It is important to note that for inhomogeneous flows this term would also include the spatial convection of the strains and thus its influence is not limited solely to unsteady flows.

Compared to the results for the cyclic shear case, the $k - \varepsilon - C_{as}$ model performs less well when comparing to the rate of decay of the RSM and $k - \varepsilon - C_{as}$ solutions. This is particularly clear from the plots of the production in the centre column of Figure 6.11, where the generation of turbulence is predicted by the $k - \varepsilon - C_{as}$ model to have all but disappeared by $t/T \sim 5$, whereas the RSM predicts that it has still not entirely disappeared by $t/T \sim 20$. In this case, the limitations of the normal stresses predicted by a linear stress-strain approximation employed in the $k - \varepsilon - C_{as}$ model are more apparent, and consequently a much lower value of the principle normal stress, $R_{11} = \overline{u_1^2}$, is predicted, which directly influences the decay rate of the production and, subsequently, of the turbulent kinetic energy and dissipation rate.

(Hadžić et al., 2001) discuss the validity of the homogenous assumption made in this calculation and state that while a weak secondary motion will usually appear as a consequence of wall friction and piston movement, this motion is generally too weak to contribute significantly to turbulence production. The authors conclude that the remaining principle sources of turbulence in an internal combustion engine are the fuel and air jets
and cavities in the piston or cylinder head, but considering the large artificial turbulence generated from EVM approaches, these models should be avoided. Therefore for an accurate computation of flows of this nature, both the Reynolds stress model and the $k-\varepsilon-C_{as}$ model appear to be suitable choices. As commented earlier, it would appear to be beneficial to incorporate a better explicit approximation of turbulent stresses into the source terms of the $C_{as}$ equation, as such a modelling scheme would be expected to offer further improvements, while remaining a computationally cheaper option to running a RSM.

The original work by Hadžić et al. (2001) was motivated by observations that a DNS of the same case also predicted that the turbulence dissapears, which lends credibility to the RSM predictions used here as a reference.
Chapter 7

Fully Developed Channel

7.1 Introduction

The fully developed channel flow has long been used as an academic test case to study the complex mechanisms that generate wall turbulence such as hairpin vortices and near wall effects (see Jimenez et al., 2004). Several experiments and a growing number of DNS calculations are available in the literature for the channel flow (see AGARD, 1997). Almost all turbulence models have been validated for this flow, and in particular it provides a means by which near wall treatment in the model can be assessed. These flows can be classified by the Reynolds number based on the wall friction velocity, \( u_\tau \), as \( Re_\tau = u_\tau \delta / \nu \) where \( \delta \) is the half channel height (useful given the symmetry of the flow) and \( \nu \) is the kinematic molecular viscosity. In the present study DNS data from Abe et al. (2001) has been used for the fully developed turbulent channel flows at \( Re_\tau = 180, 395 \) and 640. These data are more recent than the traditional data from Kim and Moin (1987) and offer a wide range of computed quantities.

The \( C_{as} \) model has been developed with bulk unsteady motion in mind, and its derivation has been based upon a high Reynolds number model for the pressure-strain term. Such a model is not well adapted for integration right down to a wall, and although low Reynolds number versions of pressure-strain models exist, they have not been implemented in this work. Thus, as indicated earlier, the terms will require some sort of special treatment to avoid over-predictions in the near-wall region. In work by Wizman et al. (1996), budgets for second moment closure schemes, obtained \textit{a-posteriori} from DNS calculations, highlight the necessity to incorporate additional modelling to capture correctly the flow physics near a solid boundary. Other work by Durbin and Pettersson-Reif
(2001) indicates that the component of the pressure-strain tensor in the wall-normal direction requires damping over a region extending to around \( y^+ \sim 100 \). The work in this chapter focuses on defining an appropriate wall treatment for the \( C_{as} \) model. The values of \( C_{as} \) calculated in the near-wall region should not degrade the performance of the underlying SST model, nor should they adversely affect the \( C_{as} \) model in the regions away from the wall.

A-priori analysis of the terms of the \( C_{as} \) equation will allow some evaluation to be made upon the effects of the different constants arising from different pressure-strain models. The effect of utilising a NLEVM approximation for the Reynolds stresses appearing in the \( C_{as} \) equation will also be evaluated, with the intention that the most appropriate set of constants will be arrived at for the continuing work on more complex applications. All the modelling developments in this chapter have been summarised in Chapter 5, although the detailed reasoning for their introduction will be given here.

### 7.2 Numerical Treatment

In all the cases presented, values are assumed as: \( \rho = 1 \), \( u_\tau = 1 \) and \( \delta = 1 \), in which case the kinematic viscosity is then taken as \( \nu = 1/Re_\tau \). Periodic boundary conditions are imposed in the stream-wise direction, with symmetry planes at the half channel height and both span-wise boundaries\(^1\). At the walls a no-slip boundary condition is applied. The computational mesh selected for this study contains 120 cells in the spanwise direction, with a near wall refinement to ensure the first point meets the criterion \( y^+ < 1 \). A view of the mesh and the layout of boundary conditions can be seen in Figures 7.1 and 7.2.

In all the cases, a constant pressure drop \( dP/dx \) is imposed as a source term in the streamwise momentum equation, where the stream-wise domain length, \( dx = 0.1 \). The constant pressure gradient can be related to the friction velocity via the overall momentum balance as follows:

\[
\frac{dP}{dx} = \frac{\rho u_\tau^2}{\delta}
\]

and so, with the previously stated assumptions, the constant pressure gradient becomes \( 1/\delta \).

\(^1\)Here, the spanwise boundaries refer to the fact that, though the problem is 2D, a ‘three-dimensional’ grid is adopted, albeit with only a single cell in the spanwise direction.
7.3 A-priori Results

From the definition of $C_{as} = -a_{ij}S_{ij}/\|S\|$, it can be deduced that its profile in a channel flow should match that of $-a_{12}$. This observation enables a simple comparison to be made and hence provides a useful study for the validation of the $C_{as}$ model. In a channel flow the transport equation for $C_{as}$ should reduce to expression 6.1, with the addition of a diffusion term. Using time averaged quantities from the DNS calculations of fully developed channel flows, the terms of the $C_{as}$ equation can be computed and, since $DC_{as}/Dt = 0$, the profile of $C_{as}$ in a channel may be approximated. The diffusion term will not be considered and thus some discrepancies can be expected. Aside from validating the derivation of the equation itself, this analysis will serve two functions. Firstly it will provide information as to the size of the near wall region where inadequate modelling of pressure-strain leads to erroneous profiles of $C_{as}$. Secondly it will provide information as to the sensitivity of the balance of the equation to the accuracy of the Reynolds stresses.

7.3.1 Tuning the Model Constants

The terms of the underlying pressure-strain models have been developed and balanced for the exact Reynolds stresses, and as such, one would expect that using an EVM to approximate these values would adversely affect the level of accuracy. In the same way, it is logical to anticipate that by using a NLEVM to represent these values, results might be improved in some cases. Figures 7.3 and 7.4 show the channel flow a-priori predictions for $C_{as}$ models based on constants from the LRR and SSG models respectively (see Table 5.1). The profiles of $C_{as}$ calculated with DNS values for turbulence quantities inserted into the source terms of the $C_{as}$ equation are compared to the DNS values for $-a_{12}$ for flows at three different Reynolds numbers.

Predictions for $C_{as}$ are arranged into three groups, based upon the method of calculating $\bar{u}_i\bar{u}_j$:

1. Using DNS values of Reynolds stresses
2. EVM approximation for Reynolds stresses (based on DNS values for velocity gradient, $k$ and $\epsilon$).
3. NLEVM approximation for Reynolds stresses based on DNS values for velocity gradient, $k$ and $\epsilon$: in this case the cubic model of Craft et al. (1996b) is employed.
In Figure 7.3, the *a-priori* calculation of $C_{as}$ using DNS values of $\overline{u_iu_j}$ agrees very well with the values of $-a_{12}$ for all Reynolds numbers, particularly near the wall, with the agreement deteriorating slightly towards the centre of the channel. When an EVM is used to provide stress approximations, the near wall profile of $C_{as}$ is largely over-predicted, and the value around $y/\delta = 0.5$, indicated by DNS values to be around 0.3 or slightly below, is predicted to be around 0.45. Use of a NLEVM for the stresses improves upon this overprediction somewhat, leading to a value of around 0.35 at the mid-region of the half channel. All three results indicate a less accurate prediction of $C_{as}$ near the centre of the channel, where turbulence quantities and the velocity gradients are small. Since diffusion becomes more significant towards the channel centreline, the inadequacies of the simple approximations taken for the diffusion term become more apparent.

Similar behaviour can be seen in the results using terms from the SSG model in Figure 7.4, although the near wall values of $C_{as}$ are slightly less accurate in all cases, as a result of the additional terms, quadratic in $a_{ij}$. The terms of the $C_{as}$ equation that are nonlinear in $a_{ij}$ have the potential to amplify errors arising though the Reynolds stress approximations. Thus it is proposed that these nonlinear terms should have a reduced influence in the $C_{as}$ model. For use with the linear EVM $C_{as}$ model, the terms $\alpha_2$ and $\alpha_3^*$ are now, therefore, set to zero, while for the NLEVM $C_{as}$ model, these terms are set to 0 and 0.1625 respectively. These proposals are summarised in Table 7.1. The current work aims to incorporate $C_{as}$ into the SST modelling framework as explained in Section 5.5. In this way the effects of $C_{as}$ will be felt only where $-0.31 \leq C_{as} \leq 0.31$ and thus it may prove beneficial to adjust the model constants to provide a value closer to the local exact definition of $C_{as} = -a_{ij}S_{ij}/\|S\|$ which, for this flow, is $-a_{12}$.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_1^*$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_3^*$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\sigma_{cas}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{as}$-EVM</td>
<td>-0.70</td>
<td>-1.90</td>
<td>0</td>
<td>0.267</td>
<td>0</td>
<td>0.75</td>
<td>1.60</td>
<td>0.2</td>
</tr>
<tr>
<td>$C_{as}$-NLEVM</td>
<td>-0.70</td>
<td>-1.90</td>
<td>0</td>
<td>0.267</td>
<td>0.1625</td>
<td>0.75</td>
<td>1.60</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 7.1: Coefficients of the $C_{as}$ equation, tuned for the linear EVM and the cubic NLEVM

Similar modifications were made to the original SSG model coefficients by Gatski and Speziale (1993) in their work on a linearised version of the SSG model. In their work, the original SSG constants $C_2$, $C_3$ and $C_3^*$ were modified to be 0, 0 and 0.18 respectively (see Table 4.3 for the original SSG model constants).
7.4 Results of Simulations

A-priori results for these modified constants are shown in Figure 7.5, and a considerable improvement is noted at all three Reynolds numbers. Figures 7.6 a and b compare the budget for the individual terms in the \( C_{as} \) transport equation 5.10 at the highest Reynolds number, \( Re_T = 640 \), for the different sets of constants. It can be seen that using the EVM approximation leads to terms 2, 3 and 4 becoming zero due to the normal stress isotropy for both sets of unmodified constants. It is this omission that is responsible for the poor prediction of \( C_{as} \)-EVM version of the model. The use of the cubic NLEVM, which provides a much more accurate prediction of the Reynolds stresses for this flow, improves the prediction of \( C_{as} \) for both the LRR and the SSG based models. In both cases, term 4, which is proportional to the expression \( a_{22} - a_{11} \) for a channel flow, is slightly underpredicted, but in fact the SSG model used alone will not return Reynolds stress predictions that match exactly the DNS values (see Figure 7.7), and so a perfect match is not expected.

Figure 7.6 c displays the corresponding budget for the modified model constants, which shows how the improved prediction of \( C_{as} \) is due in part to the reduction of term 1 to compensate for the underprediction of term 4, highlighted previously. Figure 7.6 d shows the budget on a full scale, and is displayed only for the modified constants since almost no differences are discernable between all three sets of constants for a view on this scale. However, it can be seen that the two dominant terms are terms 1 and 4, which remains the case with or without the proposed modification to \( \alpha_3^* \) in term 1.

It is important to note that improvements resulting from a modification in one case, will not, necessarily, have the same effect for another where, conversely, the same modification might worsen results. This is true for all modelling and while a-priori testing of additional flows should ultimately optimise these models, one has to remain conscious that achieving a ‘perfect’ set of constants for all cases is highly unlikely.

7.4 Results of Simulations

Simulations for the channel flow were performed using Code_Saturne, for the SSG Reynolds stress model with the scalable wall function; the SST model, and both the SST-\( C_{as} \) (EVM) and SST-\( C_{as} \) (NLEVM) schemes.
7.4.1 SSG and SST

Figure 7.7 shows the results for $Re_\tau = 640$, for the SSG and the SST models. The velocity profiles are similar in both cases, slightly below the DNS values at the centre of the channel. In the log-log plot of the velocity, the different near wall treatments are clearly visible. Below $y^+ \sim 11$ the SSG is no longer accurate, since the scalable wall function assumes $y^+ = 11$ in this region, while the SST continues to provide a good prediction. Both solutions are obtained on the same mesh (see Figure 7.1), which might not be possible using a standard wall function approach, illustrating the main advantage of the scalable wall function approach. Predictions for turbulent kinetic energy are similar for both models, although the overprediction is slightly greater from the SSG model; a fact that is likely to be linked to the choice of diffusion model used.\(^2\) In order to compare the dissipation profiles of the two models, it is necessary to convert the turbulence frequency $\omega$, as $\varepsilon = k\omega\beta_*$ where $\beta_* = 0.09$. Finally, the shear stress, in particular the peak value, is well predicted by both models, and the normal stress anisotropy returned by the RSM is clearly visible, albeit somewhat lower than that found in the DNS.

7.4.2 SST-$C_{as}$ (EVM and NLEVM versions)

Figure 7.8 shows the results for the simulations using the SST-$C_{as}$ model with both EVM and NLEVM stress approximations to provide the terms of the $C_{as}$ transport equation. It should be emphasised that this modelling framework does not use these approximated values of $\overline{u_iu_j}$ in the momentum equations, as would be done in a fully implemented NLEVM approach, although this may be the subject of future work. Results for the standard SST model are included for reference. The results are presented in columns according to Reynolds number and plots are included for mean velocity $U^+$, turbulent kinetic energy, $k^+$, turbulent viscosity, $\nu^+_t$ and $C_{as}$.

At the wall, $C_{as}$ is prescribed a value of 0.31 and this is kept constant up to the point $y^+ = 80$. This is described in Section 5.3.1, and is perhaps the simplest way to control the values of $C_{as}$ in the near wall region.

There is little difference between the model predictions for the case at $Re_\tau = 180$, where much of the flow is within the near wall region where the profile of $C_{as}$ is fixed.

\(^2\)In work for the FLOMANIA project, Uribe (2005) showed that the overprediction at the centreline of diffusion was alleviated somewhat when the tensorial diffusion model of Daly and Harlow is used.
7.4. Results of Simulations

For the cases $Re_\tau = 395$ and 640 some differences are noticed in the mean velocity profile, where the SST-$C_{as}$ (NLEVM) model predicts a slightly higher value at the centre of the channel, marginally closer to the DNS values in both cases. The prediction of turbulent kinetic energy is also slightly improved upon for the region above $y^+ = 100$, outside the region where $C_{as}$ is prescribed. These improvements are due to the modification of turbulent viscosity, as can be seen on the third row of results, where in the region $y^+ = 100 \sim 200$ the value of $C_{as}$ is such that the expression for $\nu_t$ returns a lower turbulent timescale, i.e. $C_{as}/\|S\| < C_\mu k/\varepsilon$. The profiles of $C_{as}$ confirm that the modified constants do indeed return a profile broadly in agreement with the DNS values of $-a_{12}$, with the SST-$C_{as}$ (NLEVM) performing better than the EVM variant, as expected from the a-priori results in the previous section. However, the numerical predictions fall away from the DNS values above $y^+ = 250 \sim 350$ where, as also seen in the a-priori results, the models are unable to capture correctly the behaviour of $C_{as}$ in the centre of the channel flow. The profiles of turbulent viscosity are, as a consequence, less accurate in this region. Improved modelling of the diffusion of $k$, so as to reduce the over-prediction at the channel centre, would be expected to lead to an improved prediction of $C_{as}$, and thus $\nu_t$, in the same region. The above results suggest that some improvement over the standard model predictions can be obtained with the SST-$C_{as}$, even for steady flows.
Chapter 8

Oscillating Channel

8.1 Introduction

The oscillation of flow in a pipe or channel is of high physical relevance to many turbulent flow applications; for example in engineering it appears in turbomachinery and piston engines. Such flows exhibit a number of important features such as periodic alternation between adverse and favourable pressure gradients; mean flow reversal, and inflection of the velocity profiles. Depending on the amplitude of oscillations, laminarisation of the boundary layer can occur in the acceleration phase, followed by re-transition to turbulence during the onset of the deceleration phase.

Despite its apparent simplicity, this flow is extremely challenging for URANS models. The fact that the turbulence is out of equilibrium is contrary to most of the underlying modelling assumptions, and the transition/laminarisation poses problems, since the statistical approach of one point closures often is ill-equipped to handle the dynamics of disturbances and instabilities inherent in these mechanisms. The particular case of periodically oscillating flow studied here is special in that the location of the onset of transition becomes independent of initial conditions, unlike for the flow over an aerofoil for example. The flow is characterised by the response of the turbulence, ie. the hysteresis, with respect to various flow conditions.

The Reynolds number for this flow is defined as $Re_{ls} = U_0 l_s / \nu$, where $U_0$ is a reference velocity, usually the peak value and $l_s$ is known as Stokes length, $l_s = \sqrt{2\nu/\Omega}$. In an experimental study of these flows, Hino et al. (1976) proposed a classification in terms of $Re_{ls}$ and a further parameter, $\lambda$, based on the ratio of the pipe diameter to the Stokes layer thickness, $\lambda = d/2l_s$. Their work suggested that turbulent bursting, where turbulence is
Chapter 8. Oscillating Channel

sustained over only part of a cycle, will prevail for \( \lambda > 1.6 \), and that there exist three turbulent regimes: Weakly turbulent: \( Re_l < 550 \), Conditionally turbulent: \( 550 < Re_l < 800 \) and Fully turbulent: \( 800 > Re_l \).

Experiments have shown that at low oscillation frequencies the turbulence is able to relax to a local quasi-equilibrium condition, but as the frequency is increased a phase shift is seen for certain turbulent quantities. In these latter cases, the production and dissipation rates of turbulent kinetic energy can be out of phase. At even high frequencies the unsteadiness becomes confined to the viscous sublayer, while the outer layer flow is frozen and oscillates as a plug flow.

In their work on oscillating boundary layers, Hanjalić et al. (1993) review the available experimental data and show results with a modified RSM that is capable of predicting the mean flow, and the characteristic sudden bursts of turbulence at the start of the deceleration phase. However, they state that such a model requires a certain minimum level of ‘background’ turbulence to be able to reproduce correctly its amplification, conceding that probably all turbulence models are unable to simulate the subtle details of the initiation of instability at transition for cases where \( Re_l < 800 \). In the more recent work by Cotton et al. (2001) on the modelling of periodic flows near walls, the authors conclude that while a stress-transport scheme is able to out-perform an eddy viscosity model, close attention should be paid to the modelling of diffusive transport. It is suggested that a more advanced model for the diffusion could bring about improvements in the prediction of the near-centreline region of these flows.

In an LES and DNS study, Scotti and Piomelli (2001) tested the flow for several different oscillation frequencies, and indeed show that very fine meshes are necessary in order to capture adequately the underlying flow structure. In a follow-up study by the same authors (Scotti and Piomelli, 2002), several RANS model predictions were compared to the LES results: the Spalart-Allmaras model, a low Reynolds number version of the \( k-\varepsilon \) model, the standard \( k-\omega \) model and a modified \( v^2-f \) model. Only the last of these models was able to offer adequate results, and the authors concluded that the more complex near wall treatment in that model was decisive in the finding, although modifications to the model constants were necessary to achieve satisfactory results.

In a validation of the LES implementation in Code_Saturne, Benhamadouche (2006) repeated the work of Scotti and Piomelli using the same mesh resolution, but with a shorter domain, and finite volume discretisation instead of spectral methods. This can ex-
This chapter compares results obtained using the SST-$C_{as}$ model to the reference LES results for two configurations: a low frequency case and a medium frequency case, as defined by Scotti and Piomelli. These cases were chosen to examine how well the present model is able to handle a wall bounded flow with imposed unsteadiness. The main objective here was to assess the performance of the near wall treatment. Initially, the wall treatment defined in the previous chapter is employed, where $C_{as}$ is fixed to a value of 0.31 at the wall and across the log-layer. Subsequently, a damping function is proposed, similar to the form of Van Driest damping function \(^1\) originally proposed to suppress turbulent mixing normal to the wall (Van Driest, 1956). Both of these wall treatments are summarised in Section 5.3.1.

### 8.2 Case Description

The flow is forced by a mean pressure gradient, $P_f$, defined as follows:

$$P_f(x,t) = \Delta P_0 \left[ 1 + \alpha \cos \left( \Omega t + \frac{\pi}{2} \right) \right] \frac{x}{L_x}$$

where $\alpha$ and $\Omega$ are the amplitude and frequency of oscillation respectively. The values for the two cases are presented in Table 8.1, and a comparison of the two pressure functions is shown in Figure 8.1. The pressure gradient variation adopted results in the acceleration phase beginning at $\Omega t = 2\pi/T$, where $T$ is the period time. The channel half height, $\delta$, and the mean pressure gradient, $\Delta P_0/L_x$, are set to give an average Reynolds number of 350, based on the mean friction velocity $u_\tau = \sqrt{\delta \Delta P_0 / 2 \rho L_x}$. A grid of $1 \times 120 \times 1$ nodes was used to discretise the domain for the URANS calculations with a near wall refinement, similar to that used for the fully developed channel (Figure 7.1).

The domain size for the LES calculations of Scotti and Piomelli (2001) was $6\delta \pi \times 4\delta \times 3/2\delta \pi$, while Benhamadouche (2006) used $3\delta \pi \times 2\delta \times 4/3\delta \pi$, and the number of cells adopted in both cases was $64 \times 64 \times 65$ with wall refinement. The timestep used in the calculations was adjusted so that the maximum Courant number was reported to not exceed unity. The Courant number is the ratio of the timestep, $\Delta t$, to the characteristic convection time, $u/\Delta x$, which is the time required for a disturbance to be convected a

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\(^1\)Commonly used in LES, although Van Driest originally justified its use by appeal to the oscillating Stokes boundary layer.
distance $\Delta x$.

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>$\alpha$</th>
<th>$l_x$</th>
<th>$l_x^+$</th>
<th>$Re_{l_x}$</th>
<th>$\lambda$</th>
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<td>0.10</td>
<td>35</td>
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</tr>
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<td>Medium frequency</td>
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<td>50</td>
<td>0.04</td>
<td>14</td>
<td>480</td>
</tr>
</tbody>
</table>

Table 8.1: Flow parameters for the oscillating channel

URANS calculations were made with the standard SST model and the SST-$C_{us}$ (NLEVM) with the two different wall treatments described in Section 5.3.1.

## 8.3 Results

### 8.3.1 LES Validation

Initial work was carried out to compare the two LES results for the oscillating channel flow. Since streamwise profiles were not available from Scotti and Piomelli, these comparisons were made for the periodic variation of several quantities, shown in Figures 8.2 and 8.3, for the low and medium frequency cases respectively. Overall the agreement between the two LES calculations is very reasonable, particularly for the low frequency case where the flow is more turbulent ($Re_{l_x} = 1260$) and remains so throughout the cycle. The agreement for the periodic variation of the peak energy and the wall shear stress is good, although the LES of Benhamadouche seems to overpredict the velocity in the acceleration region (between $t/T = 0$ and $t/T = 0.5$).

For the medium frequency case (Figure 8.3), the agreement is less good, where the LES of Benhamadouche always predicts higher values of both the velocity and the peak energy, especially in the deceleration phase (between $t/T = 0.5$ and $t/T = 1$). This case is the more challenging one since the Reynolds number is much lower ($Re_{l_x} = 480$) and the flow undergoes laminarisation in the deceleration period, where there is also some near-wall flow reversal (see Figure 8.7). There is a slight discrepancy in the variation of the wall shear stress between the two sets of results, where a bump appearing between $t/T = 0.6$ and $t/T = 0.8$ in the earlier simulation is not picked up by the later ones. This is mirrored by discrepancies in the periodic variation of kinetic energy over the same time bracket. This behaviour is an indication of the strong influence of near wall phenomena on the rest of the flow. These observations should be noted for the subsequent analysis and
8.3. Results

Comparison of the URANS models against the streamwise profiles of flow parameters.

8.3.2 URANS Results: Low Frequency Case

Periodic quantities

From the periodic variation shown in Figure 8.2 it can be seen that the standard SST model offers a decent prediction for all quantities until around $t/T = 0.25$, which continues until $t/T = 0.4$ for the centreline velocity. However the wall shear stress is overpredicted between $t/T = 0.25$ and 0.55, and the predictions in the deceleration phase are not as accurate. The peak turbulence energy profile is qualitatively reasonable, although the level of turbulence appears to be underestimated by a factor of about two between $t/T = 0.25$ and $t/T = 1$.

Results are also shown in Figure 8.2 for the two versions of the SST-$C_{as}$ model, differing only in how the near-wall region is treated. In the first version, $C_{as}$ is assigned a constant value of 0.31 across the near-wall layer (as described in Section 5.3.1), whilst in the second, the $C_{as}$ equation is solved across the layer, with the damping term $f_{cas}$ also described in Section 5.3.1, employed. Both versions of the SST-$C_{as}$ model give a much better prediction of the mean centreline velocity than the standard SST model, and a closer agreement with the wall shear stress and peak turbulence energy levels in the deceleration phase. The only visible difference between the two versions of wall treatment appears in the peak turbulence energy cycle, where near-wall damping offers an improvement over the standard $C_{as}$ wall treatment. The improvement is mainly seen during the deceleration phase, although the maximum value also moves forward from $t/T \sim 0.55$ to $t/T \sim 0.5$, which matches the time of the maximum value predicted by the LES results.

Phase profiles

Profiles are compared for velocity, turbulence energy and $C_{as}$ at 11 distinct times through the phase, in order to allow comparison with the LES data of Benhamadouche who provided phase averaged data at 44 such intervals. For the velocity profiles in Figure 8.4 it should be recalled that the LES of Benhamadouche appeared to overpredict the centreline velocity during the acceleration phase (from $1/11T^+$ to $5/11T^+$ in Figure 8.2). The three URANS models predict broadly similar profiles of velocity throughout the cycle, although it can be seen that the SST-$C_{as}$ models offer improved predictions over the stan-
dard SST model between \(5/11T^+\) and \(8/11T^+\). This improvement appears to occur around \(y/\delta \sim 0.2 \sim 0.3\) which corresponds to a value of \(y^+\) based on the steady wall shear stress of around \(70 \sim 100\). This region coincides with that where the SST-\(C_{as}\) was shown to offer an improvement in Chapter 7, and although the instantaneous \(y^+\) will vary according to the instantaneous wall shear stress and not its mean value, this gives an idea of the region over which the flow prediction can be expected to be affected by the introduction of \(C_{as}\).

Figure 8.5 shows the profiles of turbulent kinetic energy, and the true nature of the sudden bursting of turbulence can now be understood by the fact that there is almost zero turbulence throughout half the cycle. The appearance of much higher turbulence levels between \(4/11T^+\) and \(10/11T^+\) is captured by all three of the URANS models, although the predictions are considerably better with the SST-\(C_{as}\) model, particularly away from the wall, where the level of turbulence predicted by the SST model is too high. The benefit of applying the damping function to the \(C_{as}\) model is seen between \(5/11T^+\) and \(8/11T^+\), where the prediction of the near wall energy peak is improved.

The phase variation of \(C_{as}\) for the two different wall treatments of \(C_{as}\) is shown in 8.6, which shows that the application of the damping function corrects the near wall profiles, although the damping function appears to under-damp and over-damp \(C_{as}\) at several instances throughout the period. This observation suggests a more refined near wall formulation could be examined. The profiles of \(C_{as}\) for the standard wall treatment show that the quantity only falls below 0.3 in the region corresponding to the turbulent burst. Even without wall damping, the \(C_{as}\) model appears to be lowering the level of turbulence in response to the periodic nature of the flow.

### 8.3.3 URANS Results: Medium Frequency Case

**Periodic quantities**

The periodic results for the medium frequency case are displayed in Figure 8.3. The physics of the flow is different in this frequency range, and there is no sudden burst of turbulence energy as seen in the low frequency case. Although there is a lagging of the near wall peak turbulence behind the centreline velocity, the level of turbulence at the centre of the channel is frozen and simply advected as a plug flow.

All three URANS models offer a decent prediction of the centreline velocity, with the
results from the standard SST model being closest to the LES data of Scotti and Piomelli. The SST-$C_{as}$ model overpredicts the velocity in the acceleration phase ($t/T = 0$ to $t/T = 0.5$), although this is improved slightly when $C_{as}$ is damped near the wall, and both models give a good prediction in the deceleration phase ($t/T = 0.5$ to $t/T = 1$). The prediction of the wall shear stress in the acceleration phase is good for the three URANS models, but is less accurate in the deceleration phase. None of the turbulence models capture correctly the inflection point predicted by the reference LES data at $t/T \sim 0.65$, but the wall damped SST-$C_{as}$ model predictions are in good agreement with the LES results until this point. For the remainder of the period, the wall shear stress is underpredicted by all three models.

Prediction of the periodic variation of peak turbulent energy appears to be similar for the standard SST model and the SST-$C_{as}$ model with standard wall treatment, although this is not quite representative of streamwise profiles, as can be seen from Figure 8.8. The results from the SST-$C_{as}$ model with near wall damping show a very good prediction of the growth and decay of the peak turbulence energy level, as compared to the reference LES data.

**Phase profiles**

Figure 8.7 shows a semi-log plot of the profiles of velocity at different intervals during the cycle and, as for the low frequency case, there appears to be little difference between the URANS models. As mentioned previously, the LES of Benhamadouche overpredicts the centreline velocities when compared to the data of Scotti and Piomelli in Figure 8.3, and so one would expect the spanwise velocity profiles from Scotti and Piomelli to lie below those of Benhamadouche shown in Figure 8.7. The SST-$C_{as}$ models tend to predict higher velocities than the standard SST model towards the centre of the channel, where they also predict lower values of turbulent kinetic energy than the SST model, as seen in Figure 8.8. Once again, the influence of the $C_{as}$ modelling above a distance from the wall of around $y/\delta \sim 0.2$ is noted, roughly corresponding to the region where $C_{as}$ is allowed to modify the SST model. In addition, the effect of the wall damping of $C_{as}$ on the near wall velocity profiles can be seen for $6/11T^+$ to $8/11T^+$. Although the damping function appears to improve results up to $y/\delta \sim 0.03$, beyond this the effect is detrimental. This is perhaps further proof that although some advantages are offered by the use of this simple damping function, it is far from universally ideal.
The profiles of turbulence energy for this case are shown in Figure 8.8, where the damping of $C_{as}$ can be seen to lead to a more accurate prediction of the growth and decay of the peak energy levels. As observed in Figure 8.3 the LES of Benhamadouche overpredicts the maximum level of turbulence, particularly in the last quarter of the period. Despite this, the level of turbulence in the region away from the wall ($y/\delta > 0.2$) is still overpredicted by the SST model. Conversely the SST-$C_{as}$ models give an improved prediction of the decrease in energy levels moving towards the centre of the channel in the deceleration region (from $6/11 T^+$ to $T^+$).

The LES results in Figure 8.9 clearly depict the movement of the position of maximum $C_{as}$: towards the channel centre in the acceleration phase, and back out to the wall in the deceleration phase. The broad tendency of both SST-$C_{as}$ models to follow this motion is also evident, especially in the region $6/11 T^+$ to $T^+$, which is the same section of the period where turbulence energy is seen to grow and then decay.

8.4 Summary

In general, the SST-$C_{as}$ model offers some advantages over the standard SST model for the case of oscillating channel flows. The improvements are more significant for the low frequency case, which is perhaps due to the fact that the non-dimensional Stokes length, $l_{T^+}$, is larger in this case (see Table 8.1), as is the corresponding Reynolds number. This implies that the unsteadiness penetrates further into the flow from the wall for the low frequency case than for the medium frequency case, and as such, is present over a greater region for which $C_{as}$ is expected to offer improvements.

Although the damping function has been used here with some success, it is felt that the overdamping of $C_{as}$ seen in Figures 8.6 and 8.9 may prove to be detrimental if used for the industrial applications considered in Chapters 9 and 10. The alternative, simpler, treatment for $C_{as}$ at the wall will, therefore, be used for the work in these chapters, whereby a value of $C_{as} = 0.31$ is prescribed across the near-wall layer. It is not expected that this simplification will affect the flow to the extent reported in the present chapter, since the subsequent chapters deal with flows that are at much higher Reynolds numbers than were used in the oscillating channel flows.
Chapter 9

NACA0012 Aerofoil at 20° Incidence

9.1 Introduction

The modelling of time dependent unsteady viscous flow poses a real challenge for the aerospace industry. Turbulence modelling for unsteady aerodynamics is motivated by the fact that few existing industrial methods can always reliably predict unsteady flows whether they are separated or attached. Most computational studies to date have been concerned with the testing of algebraic, one-equation and two-equation linear EVMs, which have been shown to suppress separation and under-predict values of the unsteady loads (see Haase et al., 2003). Accurate modelling of turbulence is vital in the determination of flutter and aeroelastic forces in fluid-structure interactions, in order to simulate time dependent load fluctuations and structural response. One of the principal findings of a recent European Community project, FLOMANIA¹, which assessed the performance of a range of turbulence models, was to recommend the use of Reynolds stress transport models for industrial aerodynamic flows (Haase, 2006).

Recently the use of LES for High Reynolds number aerospace flows was addressed in the work of another European Community project LESFOIL², which aimed to assess the suitability of LES for infinite span airfoil flows, and despite highlighting the problems associated with wall treatment in LES, results were encouraging (Davidson et al., 2003). Unfortunately, this method is still prohibitively expensive as far as concerns the current industrial trend to model entire aircraft configurations.

Since most turbulence modelling methods are fundamentally based on the local equi-

¹FLOw physics Modelling, An Integrated Approach
²Large Eddy Simulation of flow around a high lift airFOIL
librium assumption for steady state flows it stands to reason that a new methodology is required in order to improve the modelling of these complex unsteady flows. A recent NASA workshop for the assessment of the direction of CFD research for the design of future generations of transport aircraft highlighted the need for further development of URANS turbulence models (Rubinstein et al., 2001).

While wind tunnel tests have long been used to measure the forces on an aerofoil in stall conditions, far fewer experimental studies have been performed that focus on obtaining experimental values in the boundary layer of a aerofoil beyond stall. One such study was performed by Favier et al. (1992) for a NACA0012 aerofoil at an incidence angle of $20^\circ$, for a Reynolds number of $1 \times 10^5$ based on the freestream velocity, $U_\infty$, and the aerofoil chord length, $c$. This study provided measurements of mean turbulent quantities in the boundary layer using an embedded Laser Doppler Anemometry (LDA) technique. This testcase was selected to assess the performance of the SST-\(C_{uu}\) model for a 2D unsteady flow of industrial relevance.

9.2 Case Description

The computational mesh used for the 2D calculations is shown in Figure 9.1. The mesh is structured, and has $\sim 2.3 \times 10^4$ cells, with a near wall refinement to ensure that the value of $y^+$ at the first node is below a value of unity. The $x$ – $y$ coordinates of the 2D domain are set to be parallel and perpendicular to the aerofoil chord in order to compare to the experimental values measured in the boundary layer. Comparisons are made in the wake of the flow, on a secondary co-ordinate system $x' – y'$ which is rotated $20^\circ$ about the $z$ axis so as to be parallel with the flow direction. A close-up view of the mesh around the aerofoil and the location of the planes extracted from the results is shown in Figure 9.2.

Inlet boundary conditions were set to match the experimental conditions, while symmetry conditions were placed at the front and back of the domain. The timestep was set to $1.0 \times 10^{-2}$ for all calculations, and the maximum Courant number was below 1 at all times.
9.3 Results: 2D Calculations

Calculations were performed using the high-Reynolds number RSM (SSG) with the scalable wall function, the standard SST model and the SST-$C_{ad}$ (NLEVM) model, hereafter referred to as the SST-$C_{ad}$ model. A periodic solution was reached in all cases after $\sim 4 \times 10^3$ timesteps, and a further $\sim 3 \times 10^4$ timesteps were required to obtain time-averaged values.

9.3.1 Pressure Forces

Surface Pressure

Figure 9.3 displays the time-averaged surface pressure coefficient $C_p$ for the experimental and numerical results. This value is obtained from the expression:

$$C_p = \frac{\bar{P} - \bar{P}_{ref}}{\frac{1}{2} \rho U_\infty^2}$$

(9.1)

where $\bar{P}$ is the time-averaged pressure at the surface, and $\bar{P}_{ref}$ is the time-averaged reference pressure. In order to compare with the experimental data the same reference pressure should be used, which in this case is the value at the leading edge of the aerofoil, and as such, all results show $C_p = 0$ at this point. The top lines represent the pressure along the upper surface of the aerofoil, where the undulations towards the trailing edge ($x/c \sim 0.5$ to 1) are a result of the flow separation, which occurs in this region. The lower lines represent the pressure along the lower surface of the aerofoil where the minimum values which occur around $x/c \sim 0.01$ are representative of the flow stagnation point.

It can be seen that the results from the RSM are vertically offset from the rest of the data. Despite this they report a profile that is in close qualitative agreement with the experimental data. This is probably due to the inaccurate prediction of the stagnation pressure at the leading edge of the aerofoil, which is used as the reference value. The use of the scalable wall-function and the associated equilibrium assumptions is most likely the cause of this discrepancy, and the use of either a low-Reynolds number RSM, or a more elaborate wall function would be expected to offer improvements.

Aside from a slight over-prediction of the lower surface pressure, the standard SST model provides a reasonable prediction up until around $x/c \sim 0.4$. The pressure on the top surface is predicted by the SST model to remain almost constant, and there is no
indication of any unsteady separation until around $x/c > 0.9$ where the pressure drops slightly. The separation region is much smaller than that indicated by the experiment and consequently the surface pressure predictions are poor in this area.

In contrast, the SST-$C_{as}$ model predicts the top surface pressure variation very well, indicating that the flow separation is well captured by the inclusion of the $C_{as}$ modifications. On the lower surface, the pressure levels are equally well predicted for the trailing edge region, $x/c > 0.7$, although at the leading edge it is slightly over-predicted, as also seen for the SST model. Even though this overprediction appears to be slight, it will have a significant effect on the prediction of the lift and drag forces acting on the aerofoil.

**Lift and Drag**

The lift and drag coefficients are obtained by integration of the pressure force around the surface of the aerofoil. The component of this force acting in the direction normal to the flow is the lift force, $L$, while the component parallel to the flow is the drag force, $D$. From these forces the non-dimensional values are obtained as:

$$C_L = \frac{L}{\frac{1}{2} \rho U_{\infty}^2 S}, \quad C_D = \frac{D}{\frac{1}{2} \rho U_{\infty}^2 A} \quad (9.2)$$

where $A$ is the plan-form area of the aerofoil, in this case equal to product of the chord length and the spanwise length of the cells = $0.05c$. The time-averaged values reported from Favier et al. (1992) are $\overline{C_L} = 0.67$ and $\overline{C_D} = 0.30$. Figure 9.4 shows the evolution of lift and drag predicted from the URANS calculations, where the time-averaged values are also plotted.

For the standard SST model there is little difference between the instantaneous force coefficients and their time-averaged values, implying that the flow is quasi-steady. Furthermore, the small unsteady motion that is present appears to be of strong periodic nature: this is a characteristic of standard URANS models which are generally unable to capture much more than the low frequency modes of unsteady motion. The additional modelling in the RSM and the SST-$C_{as}$ schemes leads to the prediction of fluctuating lift and drag forces of much higher amplitude and a somewhat less periodic signal. The time-averaged force coefficients, $\overline{C_L}$ and $\overline{C_D}$ from these two models are substantially overpredicted. The SST model does return time-averaged force coefficients that appear to be closest to the experimental values. However, having seen the poor surface pressure predictions of the SST
9.3. Results: 2D Calculations

model, this can be categorised as a fluke result, highlighting the importance of analysing the surface pressure distribution together with the lift and drag forces.

During a study of DES calculations made by Bunge et al. (2006) of the flow over the NACA0021 at $60^\circ$, it was found that the force coefficients were largely over-predicted unless the computational domain was extended in the spanwise direction to a distance of at least $4c$. For modelling frameworks such as LES and DES, and to some extent RSMs, which are able to represent unsteady structures and their interactions, it is expected that their predictive accuracy will increase if a more realistic domain is used since, after all, the experiment is not carried out on a 2D plane. Bunge et al. report that an increase in the spanwise domain from $0.4c$ to $4c$ returns an improvement in the lift and drag force prediction of almost 50%. This does give some indication that the lift and drag predictions from the SST-$C_{as}$ model in the present 2D calculations might well be reduced to be closer to the experimental data if the spanwise dimension of the calculation domain was increased.

While the standard SST has been proved to work well for attached aerodynamic flows, it is clearly unable to capture correctly the unsteady motion inherent in separated flows. In this case a quasi-steady solution is obtained due to the excessively high levels of turbulent kinetic energy and turbulent viscosity, which will be seen later on. The reasonable predictions of time-averaged lift and drag, compared to the experimental values, appear to be more of a fluke result.

9.3.2 Flow near the Aerofoil

Experimentally measured profiles of $U$, $V$ and $k$ are available at $x/c$ locations of 0.4, 0.5 and 0.6, as shown in Figure 9.2. Figure 9.5 compares these with the numerical results at each plane.

Time-averaged mean velocities

From the plots of $U$ and $V$ in Figure 9.5 it can be seen that the time-averaged mean flow shows separation in this region. Figure 9.6 displays the streamlines of the time-averaged mean flow for the three models and this confirms the existence of a large recirculation region behind the aerofoil. From the profiles, the SST-$C_{as}$ model offers the best prediction of the velocities further away from the wall, for $y/c > 0.1$, but overpredicts the magnitude
of flow reversal closer to the wall. A small additional recirculation bubble is also seen in the streamlines predicted by the SST-\(C_{as}\) model that does not appear for the other models. This small bubble is found at around \(x/c \sim 0.5\), and is picked up in the profiles of \(\overline{U}\) at the same location. It would appear that this additional recirculation region is unphysical, but in order to obtain a more conclusive understanding it would be necessary to compute a full 3D flow for this case, to examine the true nature of this structure. In general the three models deliver a reasonable prediction of the time-averaged mean velocity close to the wall, particularly the standard SST, which is somewhat surprising given the poor surface pressure predictions seen previously. However, discrepancies of this nature are understandable if one considers the non-linear relationship between velocity and pressure.

**Time-averaged turbulent kinetic energy**

The bottom row of Figure 9.5 displays a comparison of the time-averaged turbulent kinetic energy \(\overline{k}\), with experimental data. Favier et al. (1992) did not measure velocity fluctuations in the spanwise direction, and so the experimental values of \(\overline{k}\) might be slightly larger than displayed here, although the author reported that they expect that the contribution from the spanwise component to be very small compared to the two others.

The standard SST model is shown to largely overpredict the levels of \(\overline{k}\), with this over-prediction growing along the aerofoil. As one would expect, the levels of \(\overline{k}\) returned from the RSM are lower than those of the SST, and thus closer to the experimental profiles. An even better agreement is achieved by the profiles from the SST-\(C_{as}\), which agree very well with the reference data, both quantitatively and qualitatively.

**9.3.3 Flow in the Wake**

Profiles were also plotted for planes in the wake of the aerofoil, on a co-ordinate system aligned to the freestream flow direction. Although experimental data were not available at these locations (specified in Figure 9.2), informative comparisons can be made between the numerical results of the three URANS models themselves. Figure 9.7 shows profiles of the time-averaged mean velocities \(\overline{U}\) and \(\overline{V}\), and Figure 9.8 compares time-averaged mean values of turbulent viscosity \(\overline{\nu}\), turbulent kinetic energy \(\overline{k}\) and \(\overline{C_{as}}\). Planes 4, 5 and 6 are positioned at downstream distances of 0.1\(c\), 1\(c\) and 3\(c\) respectively.

Figure 9.7 shows a good agreement between the predicted time-averaged mean veloc-
9.3. Results: 2D Calculations

ities from the RSM and the SST-$C_{as}$ model. In essence, these plots offer a quantitative confirmation of the agreement seen in the streamline plots in Figure 9.6. It is the improved calculation of the production of $\bar{k}$ inherent in the RSM, and picked up using the $C_{as}$ model, which can be assumed to be the main factor here, since these models show similar results despite the use of inadequate wall treatment in the RSM framework, suggesting that the turbulence generated at the wall has little influence in the wake region.

Figure 9.8 shows the large over-predictions of $\bar{k}$ and $\overline{\nu}$ from the SST model, which are ultimately responsible for the lack of unsteadiness exhibited by this model. In Plane 5, the SST level of $\overline{\nu}$ is overpredicted by a factor of 16 relative to the RSM\(^3\), which results in a much larger diffusion of momentum, unsurprisingly damping out the majority of smaller fluctuations in the mean velocity field.

9.3.4 Analysis of the $C_{as}$ Transport Equation

The bottom row of Figure 9.8 compares the time-averaged values of $C_{as}$ from the SST-$C_{as}$ model to the values, calculated from the RSM using the definition $C_{as}^{(R)} = -a_{ij}S_{ij} / ||S||$. An agreement between these two quantities would go some way to validating the derivation and implementation of the model.

The agreement is seen to be good only in the regions where the levels of turbulence are significant, in the centre of the wake ($-0.5 < y'/c < 0.5$). Away from this region, the levels of $C_{as}$ from the SST-$C_{as}$ model are much higher than the RSM values. This is to be expected, since many of the modelling assumptions are based upon the presence of significant levels of turbulence in the flow, and the underlying NLEVM\(^4\) in particular would not be expected to return accurate predictions of the Reynolds stresses for very low levels of turbulence. It is necessary to examine the $C_{as}$ model in more detail in order to assess to what extent the accuracy of the $C_{as}$ equation is affected by the approximated values of the turbulent stress anisotropy $a_{ij}$. Despite only reporting a good prediction over a part of the flow, the previous results in this chapter can be seen as ample justification that the variation of $C_{as}$ brings an improvement to the prediction of the calculated mean flow quantities, regardless of its accuracy with respect to the values of $C_{as}^{(R)}$ calculated from the

\(^3\)The value of $\overline{\nu}$ from the RSM is obtained from the expression $\overline{\nu} = C_{as}^{2} / \bar{k}$, where the turbulent kinetic energy is obtained as $\bar{k} = 1/2(\overline{|u'|^2})$.

\(^4\)In this case, the cubic NLEVM of Craft et al. (1996b) which provides approximations of the turbulent stress anisotropy, $a_{ij}$, that are used in the source terms of the $C_{as}$ transport equation.
RSM.

Figure 9.9 shows the budget of the $C_{as}$ equation from the SST-$C_{as}$ model along Plane 5 for two snapshots of the unsteady flow, taken at times corresponding to the occurrence of maximum and minimum values of the lift force over the aerofoil (refer to equation 5.10 for numbering of the terms). Without going into great detail, it is informative to notice that large contributions to the budget appear from the convection term of the $C_{as}$ equation and Term 7, which is where the convection of the strain rate tensor $S_{ij}$ is introduced. These terms are dominant in the central wake region, $-0.5 < y'/c < 0.5$, where $C_{as}$ has already been identified to closely match the RSM values, $C_{as}^{(R)}$.

From the two snapshots it appears that these two terms oscillate throughout the unsteady cycle, and it is significant to note that they reach peak values at different $y'/c$ locations to terms 1 and 6, which are the most influential of the remaining terms. This is consistent with the idea that the convection terms are providing information about the stress transport in the surrounding flow that is unavailable at the local time and space position.

**A-priori analysis of the $C_{as}$ equation using the RSM**

It is useful to verify that the implementation of the $C_{as}$ model is correct. Figure 9.10 displays an instantaneous snapshot of the flowfield from the RSM calculation at a particular instant in time. The top plot shows the variation of the exact values, $C_{as}^{(R)}$, while the bottom plot shows the modelled value of $C_{as}$, obtained via the solution of the transport equation. The two flow fields are very similar, with regions of large positive values of $C_{as}$ corresponding to regions of large positive values of $C_{as}^{(R)}$ and the same for large negative values. It is apparent that the modelled values appear to reach slightly larger maxima and minima than the exact values, but generally the correlation is high except in the region far from the aerofoil, where the grid rapidly becomes very coarse (see figure 9.2), and numerical errors can be expected to become more apparent.

Figure 9.11 shows the instantaneous streamlines calculated using from the RSM for the same instant as used in Figure 9.10. A region of the unsteady flow in the wake has been processed to show the calculated value of $C_{as}^{(R)}$ at each cell. The eigenvalues of the turbulent stress anisotropy and strain rate tensors, $-\alpha_{ij}$ and $S_{ij}$ have then been extracted and the primary eigenvectors of each tensor plotted. In this way, values of $C_{as}^{(R)}$ can be compared to the physical alignment of the two tensors.
Since the eigenvectors for a $3 \times 3$ symmetric tensor are orthogonal, the minimum alignment will occur when the principle eigenvectors of $S_{ij}$ and $-a_{ij}$ are positioned at an angle of $45^\circ$ to each other. This configuration corresponds to the regions of $C_{as}^{(R)} \sim 0$. The maximum positive and negative values of $C_{as}^{(R)}$ occur when the two tensors are fully co-aligned: for the maximum value of $C_{as}^{(R)}$ all aligned eigenvectors will have the same sign, and for the minimum value of $C_{as}^{(R)}$ the aligned eigenvectors will be of opposing sign. The distinction between maxima and minima cannot be made without knowing the sign of the eigenvectors, and since only the principle eigenvectors were plotted in Figure 9.11, it is not possible to see this in detail. However the rotation of the principle axis of the two tensors can clearly be seen: for values of $C_{as}^{(R)} \sim 0.577$ they are mutually perpendicular, and for values of $C_{as}^{(R)} \sim -0.577$ they are parallel.

This simple analysis of eigenvector alignments is, of course, only possible given the 2D nature of this calculation and these generalisations would not necessarily be the same for a fully 3D flow. This comparison is thus included to indicate the physical relevance of $C_{as}$, in order to justify the modifications it makes to the predicted flowfield, which have been seen to lead to improvements over the standard SST model.
Chapter 10

Circular Cylinder in Square Duct

10.1 Introduction

The accurate prediction of turbulence quantities in the wake of flows past bluff bodies is of significant interest to industry, and has been an important component of turbulence model validation for some time. The turbulent flow around a circular cylinder has become perhaps the most widely tested bluff body flow since the experimental work of Cantwell and Cole (1983), who studied this geometry in an unconfined flow\(^1\) at a Reynolds number of $1.4 \times 10^6$ based on the cylinder diameter and free stream velocity. This case was originally selected for inclusion in the present study in order to examine the physics of coherent turbulent structures, since the regular pattern of vortices shed behind the cylinder, known as the von Kármán street, remains clearly distinguishable for the first few diameters downstream of the cylinder.

The interaction of the shear layers with the vortex street results in a roll up process, producing a stable, slowly changing wake. The clearly defined point of separation, and the unsteady vortex interactions, lend themselves to being an attractive test case for hybrid modelling approaches such as DES, which is well suited to such cases since the ‘detached’ regions are easily defined (see Travin et al., 2000).

The resolution of the near wake flow in the work of Cantwell and Cole (1983) has become somewhat inadequate for the validation needs of more advanced turbulence models and finer computational meshes. To remedy this, more recent work by Djeridi et al. (2002) and then Perrin et al. (2005a) repeated the experiment for a circular cylinder in a

\(^1\)For the experimental configuration of Cantwell and Cole, the test section was located at a distance of around $15D$ from the sides of the wind tunnel walls, where $D$ is the diameter of the cylinder.
square duct at the same Reynolds number used by Cantwell and Cole. The flow structure was analysed over a refined grid of measurement points, using Laser Doppler Velocimetry (LDV) and particle image velocimetry (PIV) techniques.

The experimental work of Perrin et al. (2005a) is used in the current study. This chapter presents the results of calculations made for the same geometry and flow conditions for the same three models as in the previous chapter: the RSM (SSG), the SST model and the new three-equation SST-$C_{as}$ model developed in this thesis. Calculations are first shown for a 2D flow, corresponding to a plane at the centre of the square duct, thereby neglecting the influence of the spanwise walls. Finally, in order to assess the performance of the SST-$C_{as}$ model for a 3D case, the full geometry is computed and preliminary results are shown.

### 10.2 Case Details: 2D Calculations

In the experiment of Perrin et al. (2005a), the aspect ratio was set to be $L/D = 4.8$ corresponding to a blockage ratio of $D/L = 0.208$, where $L$ is the side of the square duct and $D$ is the diameter of the cylinder, and the level of freestream turbulence intensity was reported to be 1.6%. A schematic of the experimental setup is shown in Figure 10.1, and the computational mesh used for the 2D calculations is shown in Figure 10.2, which also displays the relative location of the two planes over which PIV data was obtained. The mesh is structured, and has $\sim 5.3 \times 10^4$ cells, with a near wall refinement to ensure that the value of $y^+$ at the first node is at a maximum of 1. A higher concentration of cells was arranged behind the cylinder in the region of the near wake flow. The coordinates of the 2D domain are set to correspond to the $x-y$ plane at the location $z/D = 0$, which corresponds to the location of the PIV planes.

Inlet boundary conditions were set to match the experimental conditions, while symmetry conditions were placed at the front and back of the domain. The timestep was set to $1.0 \times 10^{-2}$ for all calculations, and the maximum Courant number was below a value of unity at all times.
10.3 Results: 2D Calculations

Calculations were obtained for the high-Reynolds number RSM (SSG) with the scalable wall function, the standard SST model and the SST-\( C_{as} \) model. Time-averaging was begun after \( \sim 1 \times 10^4 \) timesteps in all cases, and a further \( \sim 6.5 \times 10^4 \) timesteps were required to obtain time-averaged values.

10.3.1 Surface Pressure Measurements

Experimentally measured values of the nondimensional surface pressure around the surface of the cylinder are available at the spanwise location \( z/D = 0 \). A comparison with the numerical predictions is shown in Figure 10.3, where \( \theta = 0^\circ \) corresponds to the forward stagnation point.

All results are very similar up until around \( \theta = 50^\circ \), from which point the URANS models predict a deeper \( C_p \) compared to the experiment, indicating a higher level of turbulence and a prolonged region of attached flow around the cylinder. The point at which the pressure reaches a plateau is usually indicative of the separation point, denoted \( \theta_s \). In this way, and with reference to the mean velocity fields, the location of the predicted separation points are indicated on Figure 10.3 for each model. The RSM is seen to predict separation at around \( \theta_s = 120^\circ \), while the SST model indicates a separation point around \( \theta_s = 111^\circ \) and the SST-\( C_{as} \) model predicts the slightly earlier location of around \( \theta_s = 108^\circ \). Perrin et al. (2005b) reports that the Reynolds number for this flow lies within the critical range of values between laminar and fully turbulent flows, which might explain why the turbulence models, which are not ideally suited to handle transition, predict a more turbulent separation in this case. The separation point reported in the experiment was close to \( \theta_s = 80^\circ \), which is consistent with a laminar separation.

The SST-\( C_{as} \) model returns the closest prediction of \( \overline{C_p} \) to the experimental data in the region \( \theta = 50^\circ - 120^\circ \), significantly better than the SST model, with the RSM returning the poorest prediction. In the region behind the cylinder, both the SST and the SST-\( C_{as} \) model provide a good prediction of the level of \( \overline{C_p} \), which in turn, indicates a reasonable prediction of the drag force. The RSM predicts a much higher value of \( \overline{C_p} \) behind the cylinder due to the use of the scalable wall function, where the assumption of a log-law profile is far from ideal for this region. A more accurate prediction would be expected with the use of a low-Reynolds-number RSM framework, or a more advanced wall-function
scheme that is able to account for the non-equilibrium conditions in the separated region.

### 10.3.2 PIV Spectra

Temporal PIV data is also available from the experiment for several locations in the near wake flow of the cylinder. Figure 10.4 presents a comparison of the spectra of the $U$ velocity signals at 5 points, the location of which are defined in the figure. A fast-fourier-transform (FFT) has been applied to the time series, which for the numerical results contained a set of $2^{16}$ points (see Frigo and Johnson, 1998, for details of the algorithm used.).

The plots in Figure 10.4 show the calculated spectra for the experimental data and the 3 URANS models used. The line $E(U) = f^{-5/3}$, is also plotted to show the characteristic gradient of equilibrium turbulence. It can be seen that the gradient of the experimental data is considerably less than that of $f^{-5/3}$, a clear indicator that the turbulence is not in a state of energetic equilibrium at these locations. For the numerical results at low frequencies ($f < 0.2$), the spectra returned by the SST-$C_{as}$ model is closer to the experimental levels than the other two URANS models. The SST-$C_{as}$ model also appears to give the best prediction of the energy levels in general, where the slope of the spectra are closer to experimental values. This is particularly true for point 1, in the shear layer of the wake, where the SST and the RSM models predict a gradient close to the equilibrium value of $f^{-5/3}$.

The peak value of the spectra corresponds to the natural frequency of the flow, the shedding of the Von Kármán vortices. This frequency is shown to be around 0.22 for the experimental data, while the URANS model predict slightly higher values of around 0.26 for the SST model and 0.29 for both the RSM and SST-$C_{as}$ models. This discrepancy may be attributed to the 2D nature of the calculation, which neglects the effects of the walls at $z/D \pm 2.4$. The fact that the SST appears to return a peak frequency closer to the experimental value may thus be discounted to some extent, since the results from a fully 3D calculation may well reduce this frequency beyond the experimental value. The agreement between the RSM and the SST-$C_{as}$ model is encouraging, as this is one of the main objectives of the $C_{as}$ modelling approach.

All three URANS models indicate several subsequent energy peaks that are not shown in the experimental results, the first three of which are seen at frequencies around $f = 0.56$, 0.90 and 1.20. This is a further consequence of the unphysical two dimensionality of the flow in these calculations, which allows the eddies in the wake to amalgamate and
10.3. Results: 2D Calculations

Instabilities in the spanwise direction of the real flow tend to interact with and break down these eddies, and thus only the primary peak remains. However it can be noted that at the higher frequencies \( f > 1.6 \), particularly for the points directly behind the cylinder, the SST-\( C_{as} \) model predicts a more uniform distribution of energy than the other two URANS models, indicating that the smaller structures are more represented in a more physically correct manner with the SST-\( C_{as} \) model.

10.3.3 PIV Planes

Phase-Averaging

The phase-averaging is first described in order to enable an accurate comparison with numerical results. According to the method outlined by Hussain and Reynolds (1970), the instantaneous velocity, \( \tilde{U}_i \), is classically decomposed as:

\[
\tilde{U}_i = \overline{U}_i + U'_i + u_i
\]

comprising a time-averaged component, a periodic fluctuation and a random fluctuation in that order. The phase averaged velocity is then defined as the sum of the time-averaged and periodic fluctuations, \( \langle U_i \rangle = \overline{U}_i + U'_i \). It can be deduced from equation 10.1 that the phase-averaged Reynolds stresses, \( \langle U_i U_j \rangle \), will be the sum of the average of the product of the periodic fluctuations, \( \overline{U'_i U'_j} \), and the average of the product of the random fluctuations, \( \overline{u_i u_j} \) as shown:

\[
\langle U_i U_j \rangle = \overline{U'_i U'_j} + \overline{u_i u_j}
\]

The quantity \( \overline{U'_i U'_j} \) can be expected to be much larger than \( \overline{u_i u_j} \) in the near wake flow due to the unsteady periodic nature of the flow.

Mean velocity field

Figure 10.5 displays a direct comparison of the mean velocity components \( \overline{U} \) (left column) and \( \overline{V} \) (right column), between experimental values and computed values from the three URANS models: the RSM, the standard SST and the SST-\( C_{as} \). For each figure the area above the centreline displays the experimental results and the area below the centreline displays the numerical results, while the scales remain the same and contour lines are included for clarity. In the plots of \( \overline{U} \) a line is drawn from the centre of the cylinder to the
point of separation on the cylinder surface: a red line for the experiment and a black line for the numerical results. It should be noted that the vertical mean velocity component, \( \overline{V} \), is anti-symmetric about the centreline \( Y = 0 \), but here these figures employ a mirror symmetry for ease of comparison. The left column of Figure 10.6 displays a comparison of the mean flow streamlines for the experiment and the URANS models.

From the results for the high-Reynolds number RSM (SSG) with scalable wall function, it can be seen that the recirculation region is considerably smaller than in the experiment. Comparison of the streamline plots indicates that the recirculation region is around 44% of the length indicated from the experiment. The flow above and below the cylinder in the region of the separation point is also somewhat different. The contour \( U = 1.45 \) for the RSM extends further away from the cylinder in the wall normal direction than in the experiment, which is consistent with a more turbulent boundary layer predicted by the RSM.

The results from the SST model predict an earlier point of separation than the RSM, but the recirculation region is even smaller than that predicted by the RSM: around 25% of the length of the experimental region. The closest prediction is returned by the SST-C_{as} model, which gives an accurate representation of the shape of the mean recirculation region, and a recirculation length of around 81% of the experimental value. The streamlines of the SST-C_{as} scheme indicate what appears to be smaller secondary recirculation near the point of flow separation, which is not indicated in the experimental streamlines. It is possible that a similar perhaps slightly smaller recirculation does exist in the experiment, but is not captured. Closer examination of the contour \( U = 0.05 \) in the same region for both the RSM and the SST-C_{as} model indicate a slight kink which may be a sign that a small secondary recirculation does exist with both models.

**Turbulent kinetic energy levels**

Figure 10.6 displays the phase-averaged values of turbulent kinetic energy from the experiment, \( \langle K \rangle = \frac{1}{2} \langle u_i u_i \rangle \), which is the sum of the the contribution from periodic fluctuations and random fluctuations, as described in equation 10.2. The numerical results represent only the time-average of the random component, \( \overline{k} \), of this sum and as such these levels are expected to be less than the time-average of the phase-averaged quantity \( \langle K \rangle \), particularly since the periodic fluctuations are expected to be much larger than the random fluctuations. A direct comparison of values is therefore not possible without an
approximation for the mean of the periodic fluctuations, which was not obtained in this work. However, this comparison allows the relative magnitude of the periodic and random fluctuations to be seen, and also allows the performance of the URANS models to be assessed with respect to each other.

The levels of $\bar{k}$ are very similar for the RSM and the SST-$C_{as}$, and the peak values occur in the region immediately behind the cylinder, close to the separation point. Indeed it is this region where one would expect the random fluctuations to dominate over any coherent periodic motion, which is reinforced by the experimental values of $\langle K \rangle$ in the same region, which are close to zero. In contrast, the levels of $\bar{k}$ from the standard SST model are much higher in the wake of the cylinder. In particular, the levels indicated for the small region near the separation point are in some cases even higher than the levels of $\langle K \rangle$ from the experiment, indicating a significant overprediction, which is consistent with the subsequent underprediction of the separation region behind the cylinder.

Overall the SST-$C_{as}$ model provides the most accurate prediction of the flow for the 2D analysis, superior to the prediction of the standard SST model. Final, definitive, conclusions about the RSM model cannot be drawn here since the use of a more suitable near-wall treatment can be expected to improve its performance. Consideration of the fully 3D flow can also be expected to improve predictions for all models, and this is now addressed in the following section.

## 10.4 Case Details: 3D Calculations

The computational mesh used for the 3D calculations is shown from several angles in Figure 10.7. The mesh is structured, and has $\sim 2 \times 10^6$ cells, with a near wall refinement to ensure that the value of $y^+$ at the first node is at a maximum of 1. A higher concentration of cells was arranged behind the cylinder in the region of the near wake flow.

Inlet boundary conditions were set to match the experimental conditions, and no-slip boundary conditions were defined at the walls. The timestep was set to 0.01 for all calculations, and the maximum Courant number was below 1 at all times.
10.5 Results: 3D Calculations

Calculations were obtained for the high-Reynolds number RSM (SSG) with the scalable wall function, the standard SST model and the SST-\(C_{as}\) model. Time-averaging was started in all cases after \(\sim 1.6 \times 10^5\) timesteps, and a further \(\sim 2.5 \times 10^5\) timesteps were performed in order to obtain the time-averaged quantities. While the averaging time was deemed to be adequate for the time-averaged flow quantities, it was of inadequate length for an accurate spectral analysis of the unsteady velocity, of the type performed for the 2D calculations in Section 10.3.2.

10.5.1 Surface Pressure Measurements

Experimental values of the nondimensional surface pressure around the surface of the cylinder are available at the spanwise location \(z/D = 0\). A comparison with the numerical predictions is shown in Figure 10.8, where \(\theta = 0^\circ\) corresponds to the forward stagnation point.

Results are very similar to the surface pressure plots in the 2D analysis discussed in Section 10.3.1 and shown in Figure 10.3. As expected, the prediction of the separation point, \(\theta_s\), is improved slightly in the 3D analysis: the RSM results predict separation at \(\theta_s = 112^\circ\), an improvement of \(8^\circ\) over the 2D calculations, while the SST model indicates a separation point of \(\theta_s = 107^\circ\) and the SST-\(C_{as}\) model predicts \(\theta_s = 103^\circ\), bringing improvements of \(4^\circ\) and \(5^\circ\) respectively, over the 2D calculations. Despite these improvements, the values are still significantly behind the experimental separation of around \(\theta_s = 80^\circ\).

In the region \(\theta = 50^\circ - 120^\circ\), the results of the standard SST model and the SST-\(C_{as}\) model are similar, with the latter model returning the slightly more accurate prediction, and the RSM returning the poorest prediction.

10.5.2 PIV Planes

Mean velocity field

Figures 10.9 and 10.10 are displayed in the same manner as Figures 10.5 and 10.6 from the 2D analysis, the layout of which has been described in Section 10.3.3.

Whereas in the 2D calculations it was seen that RSM under-predicted the recirculation
region, here it can be seen that it is overpredicted by around 60%. With reference to the contours of $\bar{V}$ for the RSM in Figure 10.9 it can be seen that the vertical mean flow immediately behind the cylinder is somewhat poorly predicted. In fact the vertical mean flow is shown to be weak for a distance extending almost $1D$ behind the cylinder, whereas the PIV data describes a more compact recirculation with higher velocities.

More accurate predictions of the mean flow are obtained from the standard SST model and the SST-$C_{as}$ model; predicting recirculation lengths of 80% and 88% respectively of the experimental value. The contour levels of $\bar{U}$ and $\bar{V}$ are also much closer to the PIV data, with slightly higher velocities seen from the SST model. The secondary recirculation bubble seen in the 2D analysis with the SST-$C_{as}$ model is not seen here, implying that this was a feature specific to the 2D case, although it should be noted that less time-averaging has been performed for the 3D calculations.

Comparison of the velocities predicted by the SST and SST-$C_{as}$ models in the region below the cylinder, near the separation point indicates a slight improvement from the SST-$C_{as}$ model relative to the experimental contours. This is consistent with the improved surface pressure distribution discussed previously.

**Turbulent kinetic energy levels**

Figure 10.10 displays the phase-averaged values of turbulent kinetic energy from the experiment, and the numerical predictions of the random component, $\bar{k}$, as described in Section 10.3.3 for the 2D analysis.

While the peak levels of $\bar{k}$ predicted by the SST model again reach a high level, with a maximum of $\bar{k} = 0.08$ behind the cylinder, the peak levels predicted by the SST-$C_{as}$ model and the RSM are similar to each other. The region defined by the contour $\bar{k} = 0.03$ extends further downstream from the back of the cylinder for the RSM than for the SST-$C_{as}$ model, although as already observed, so does the recirculation region for the same model. The turbulence arising from random fluctuations is likely to reach its peak downstream value immediately behind the cylinder, as seen in the results of the SST and SST-$C_{as}$ models, since the flow in this region is largely unaffected by the periodic motion which otherwise dominates the energy spectra. The absence of large values of $\bar{k}$ close to the cylinder returned from the RSM predictions is perhaps attributable to the use of a standard wall function approach. The delayed separation of the RSM has the effect of reducing the spreading of the wake downstream of the cylinder. This can be seen in both Figures 10.9
and 10.10 where the contours of $\bar{U}$ and $\bar{k}$ remain parallel to the flow direction for a greater downstream distance: around $1D$ compared with less than $0.5D$ for the SST and SST-$C_{as}$. This reduced spreading is likely to be evident also in the downstream evolution of turbulent structures.

10.5.3 Flow Visualisation of Coherent Structures

For the 3D analysis, the coherent structures that form in the wake of the cylinder and their interactions can be computed and compared to experimental observations.

Streamwise structures

Djeridi et al. (2002) performed visualisations for this flow using a cavitating regime, a snapshot of which is shown in Figure 10.11. The von Kármán vortices are clearly visible, as are the smaller scale streamwise vortices shown between them. The experimenters reported a slow, unsteady oscillation of the vortex street in the wake of the cylinder.

Figure 10.12 displays plots of instantaneous values of the magnitude of the vorticity, $\omega_i$, which is defined as:

$$\omega_i = \{\omega_x, \omega_y, \omega_z\} = \left\{ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial z} - \frac{\partial u}{\partial y} \right\}$$

for the three URANS models at three successive instances during the time-averaging phase of the calculation. The snapshots are not shown at regular phase intervals but nevertheless give an indication of the nature of the evolution of structures in the wake, and plots are shown in greyscale for comparison with the photo from Djeridi et al. (2002). Perhaps unsurprisingly, the least unsteady structures appear in the wake of the calculations using the standard SST model. In these frames the vortices appear to move in a line directly behind the cylinder with little of the vertical sweeping motion reported in the experiment. While there does seem to be some streamwise vortices between the primary structures, they are not particularly well defined. The wake from the RSM calculation displays some vertical displacement of the primary structures in the wake, but there also appears to be a distinct lack of structures forming in the immediate vicinity of the cylinder, likely due to the poor predictive nature of the scalable wall function for the non-equilibrium flow in this region.

The snapshots from the calculation where the SST-$C_{as}$ model was used indicate a more
highly convoluted wake topology, with the vertical variation of structures and the smaller streamwise structures both clearly visible. These streamwise structures, that can be seen in the form of dark curves linking together the main structures, are seen to appear in pairs, as also indicated from the photo in Figure 10.11.

Vortex dislocations

An earlier study from Williamson (1992) for the unconfined flow around cylinders at lower Reynolds numbers reported the phenomenon of vortex dislocation, which describes the tendency of the von Kármán vortices to bunch up in the spanwise direction, forming ‘hairpin-like’ structures in the wake of the cylinder. This can be seen in Figure 10.13, which views the wake from above, with the cylinder located at the left-hand-side of the photo and the flow direction from left to right. These ‘hairpin’ structures can be seen forming in the second and third rows of the vortex street, and this process is seen quite clearly due to the low Reynolds number of the flow.

In order to draw a comparison from the numerical results, the structure parameter, \( Q \), is calculated, which was found by Hunt et al. (1988) to be an effective way to visualise the regions of coherent vorticity due to rotational motion (as opposed to those from shear), and is defined as:

\[
Q = -\frac{1}{2} \left( S_{ij}S_{ij} - \Omega_{ij}\Omega_{ij} \right)
\]  

(10.4)

where \( Q \) should take a positive value. Figure 10.14 shows instantaneous iso-surfaces of \( Q = 0.5 \) for the three URANS models, viewed from above for comparison with Figure 10.13 and from an oblique angle. The lack of coherent structures is apparent in the results from the SST model, for all the wake region except very close to the cylinder and the side walls. For this model it is the over-predicted values of turbulent kinetic energy, leading to high values of the turbulent viscosity, that are responsible for the early damping out of these structures. As a result the vortex dislocations are not seen in the SST results, whereas those from the RSM clearly show the ‘hairpin-like’ structures reported by Williamson (1992). The lack of structures in the immediate vicinity of the cylinder for the RSM results is seen clearly from the oblique angle, whereas both the standard SST and the SST-\( C_{as} \) model show more structures in this region. The vortex dislocations can also be seen in the results from the SST-\( C_{as} \) model and there are significantly more spanwise structures than seen with either of the two other models. In fact, the wake predicted by
the SST-$C_{as}$ scheme displays much more structures than seen in the experimental snapshot, but then this flow is at a much higher Reynolds number and this might, therefore, be expected.

The reduction of the turbulent kinetic energy brought about by the action of the $C_{as}$ term can clearly be seen to prevent the premature damping of coherent structures. While the level of unsteadiness is far from what would be expected from a Large Eddy Simulation, or a hybrid approach such as DES, the cost in terms of computational time and grid requirement is much less for the SST-$C_{as}$ model, and the reduction of viscosity is brought about in a physically justifiable manner, which could be exploited in a DES framework or a hybrid RANS-LES approach.
Chapter 11

Conclusions and future work

11.1 Conclusions

This thesis has described the development and validation of a new turbulence model that is sensitive to the local stress-strain lag in mean unsteady turbulent flows. A parameter which is representative of this feature has been defined as $C_{as} = -a_{ij}S_{ij} / \|S\|$, where $a_{ij}$ is the turbulent stress anisotropy tensor; $S_{ij}$ is the strain rate tensor, and $\|S\|$ is an invariant defined as $\sqrt{2S_{ij}S_{ij}}$.

Following observations from Hadžić et al. (2001), who highlighted the modelling deficiencies of the standard eddy viscosity formulation for a homogenous cyclic strain flow, it has been reasoned that solving a transport equation for $C_{as}$, and introducing this into the expression for the production rate of turbulent kinetic energy will provide a more accurate representation of the latter. By doing this, information concerning the transport and history effects of the turbulent stresses, which would not be available in a standard two equation scheme, can be expected to influence the predicted levels of turbulent kinetic energy in such a way as is seen when using a full Reynolds stress transport model (RSM).

During preliminary work, the $C_{as}$ model was implemented into a simple 1D code for the homogenous strain calculations presented in Chapter 6, and was used with a standard $k - \varepsilon$ model framework, referred to as the $k - \varepsilon - C_{as}$ model. In all the subsequent results Chapters, the transport equation for $C_{as}$ was implemented into a 3D unstructured finite volume code, Code_Saturne, where it has been used together with the Shear-Stress Transport model (SST) of Menter (1994), in what is referred to as the three equation SST-$C_{as}$ model. Results have already been discussed on a case-by-case basis in each Chapter, but salient points are repeated here in order to emphasise the principal conclusions.
Initial response to imposed shear

- When a shear is applied to homogenous isotropic turbulence, the Reynolds stress transport model predicts that the turbulent shear stress anisotropy, $a_{12}$, evolves from an initial value of zero, whereas both the linear and cubic $k - \varepsilon$ models predict a non-zero initial value. The $k - \varepsilon - C_{as}$ model is shown to respond to an applied shear in a similar way to the RSM, since in this case it effectively acts as a transport equation for $a_{12}$.

Time dependent cyclic strain

- In both the cyclic shear and cyclic strain flows, a stark difference was observed between the results of the Reynolds stress transport model, which indicate a decay of turbulent kinetic energy, and those of the linear (and cubic) $k - \varepsilon$ models, which both predict an exponential growth of the same quantity.

- The $k - \varepsilon - C_{as}$ model is able to reproduce the decay seen when using the RSM by calculating a variation of $C_{as}$ that oscillates about zero, thereby bringing a zero net production of turbulent kinetic energy to the flow. The influential term is the rate of change of strain term, which forces the production rate to become negative midway through the cycle.

Fully developed channel

- The transport equation for $C_{as}$ is shown to be sensitive to the accuracy of the Reynolds stresses, $\overline{u_i u_j}$, which are used in the calculation of the source terms of the $C_{as}$ equation (as opposed to the representation of the stresses used in the momentum equations). The cubic NLEVM of Craft et al. (1996b) was selected to approximate $\overline{u_i u_j}$ for the terms of the $C_{as}$ equation, and the model coefficients have been tuned for a fully developed channel flow at different Reynolds numbers.

- A near wall treatment for $C_{as}$ has also been developed, whereby the value of $C_{as}$ is set to 0.31 at the wall, and is fixed at this value across the log-layer, from the wall up to a distance of $y^+ = 80$. 
11.1. Conclusions

Oscillating channel

- The SST-$C_{as}$ model is shown to improve predictions of the periodic behaviour of turbulent quantities relative to the standard SST model. The streamwise profiles of $C_{as}$ are shown to oscillate in a manner comparable to LES reference data of the same flow.

- An alternative near wall treatment has also been proposed and tested in this flow, in which a damping function tuned to channel flows, is applied to the source terms of the $C_{as}$ equation.

- Although the above near-wall treatment does allow some of the benefits of the $C_{as}$ model to be introduced into the near-wall region, it was felt that the damping cannot be used reliably for more complex cases, where over-damping might have an adverse effect on the turbulent viscosity. Hence the simpler, previously described, near-wall treatment was used for the remainder of the calculations in this thesis, (i.e. the value of $C_{as}$ is fixed at the wall and across the log-layer).

NACA0012 aerofoil at 20° incidence

- The SST-$C_{as}$ model predictions have been compared to those of the standard SST model for the flow around an aerofoil beyond stall. Surface pressure predictions have been shown to be more accurate when the SST-$C_{as}$ model is used, as are the time averaged velocity and turbulent kinetic energy predictions in the boundary layer and in the wake of the aerofoil.

- *A-priori* analysis of the $C_{as}$ equation has confirmed the correlation between the scalar value of $C_{as}$ and the physical mis-alignment of the tensors $S_{ij}$ and $a_{ij}$. However, this finding does not automatically infer the same conclusions for 3D flows, where more investigation may be required because of the extra complexity of defining and characterising the mis-alignment of tensors.

Circular cylinder in square duct

- The SST-$C_{as}$ model is shown to return improved predictions of the time-averaged velocity field in the wake of the cylinder, relative to the standard SST model and the experimental data. The surface pressure distribution predicted by the SST-$C_{as}$
model is also improved, and the location of the separation point is more accurately predicted with respect to the experimental value, although it should be noted that all URANS models were unable to exactly capture the early laminar separation reported in the experimental work on this case, which is carried out at a Reynolds number in the ‘critical’ regime between purely laminar separation and purely turbulent separation.

- The use of the scalable wall function with a high-Reynolds number Reynolds stress transport model is inappropriate given the non-equilibrium nature of the flow in the wake of the cylinder.

- A spectral analysis of the experimental velocity fluctuations in the wake of the cylinder indicates that the energy transfer is at a slower rate than the equilibrium $-5/3$ cascade. While the results from the SST-$C_{as}$ model were seen to be broadly in line with these experimental observations, the standard SST model was unable to reproduce this behaviour.

- The 2D simplification of this flow was shown to lead to additional coherent structures in the wake that would otherwise be destroyed by the interaction of streamwise structures that would appear in the fully 3D flow.

- The SST-$C_{as}$ model predicts the evolution of turbulent structures that appear to be less periodic that those obtained from the standard SST model. The SST model in general is seen to damp out the majority of coherent structures in the wake of the cylinder, whilst the structures predicted by the SST-$C_{as}$ model have been shown to resemble qualitatively the visualisations from the experimental work on this flow.

11.2 Suggestions for Future Work

This work has made some significant progress towards the original objectives outlined in the Introduction, but it represents only the start of the development and validation process that would be necessary in order to confidently apply a generic form of this model to a broad range of test cases. Certain elements of the modelling described in this thesis were, of necessity, not explored as fully as they could have been, in order to implement and test a robust scheme within the given time frame. There thus remain several areas within which further work would appear to be beneficial:
• **Near wall modelling:** One of the most significant modelling steps taken in this work has been the development of the $C_{as}$ transport equation based on a high-Reynolds number stress transport model. However, this does mean the resulting model inherits the shortfalls of the underlying high-Reynolds number RSM if applied in the near wall region. Two near wall treatments for the $C_{as}$ model were proposed in this work, but ultimately, only the simple method of prescribing a fixed value of $C_{as}$ across the near-wall layer has been used with confidence. Future work could focus on the near-wall modelling of the $C_{as}$ equation, to account properly for viscous and wall proximity effects.

• **Selection of Reynolds stress approximation:** In this work the $C_{as}$ model has been developed and tuned with the cubic NLEVM of Craft et al. (1996b) providing a representation for the stresses appearing in the $C_{as}$ equation. This could be substituted for any other similar scheme that provides the Reynolds stress approximations via algebraic expressions in terms of the mean flow quantities. The present cubic model was selected in light of the reasonable success it has seen across a range of flows, and its relatively compact form, which is an advantage in the implementation process. Indeed, a new non-linear set of algebraic equations could be derived specifically for the $C_{as}$ framework, and tuned with unsteady mean flows in mind, or, with the above comments on near-wall modelling in mind, it might be worthwhile to explore the use of NLEVM’s which give a better representation of the near-wall stresses. It would also appear to be advantageous to formulate such a model in a way which includes the estimated Reynolds stresses in the momentum equations, so as to benefit from the influence of improved estimates of these on the mean flow, as is required to correctly resolve the secondary motion in the corners of a square duct flow, for example (see Huser et al., 1994, for more details).

• **Tuning of model coefficients:** In this work, the model coefficients of the source terms in the $C_{as}$ equation are tuned with respect to DNS analysis of a fully developed channel, and might therefore require additional tuning for flows of significantly different character. The cases of streamline curvature, swirl and impinging flows might be amongst some other cases that are relevant for model tuning. It should be noted that, in principle, the model coefficients of the underlying pressure-strain model, as well as the coefficients of the employed NLEVM, will normally
have been tuned for some of these cases, but it remains the case that additional modifications, when implemented within the $C_{as}$ equation, may be necessary.

- **Further analysis of the physical interpretation of $C_{as}$ for 3D flows:** The misalignment of the two tensors in a 2D flow can be defined by a single angle and it is therefore relatively easy to draw comparisons between the value of $C_{as}$ and the physical state of alignment, as has been shown in this thesis. However, three independent angles are required to define the relative tensorial alignment in a fully 3D flow, and as such, it remains unclear as to whether or not the returned value of $C_{as}$ for a given state of alignment would correlate correctly to the physical situation.

- **Application to statistically steady flows:** While the modelling work in this thesis was not intended to influence the results of statistically steady flows, some observations in the work on the channel flow in Chapter 7 suggest that the $C_{as}$ model may offer some improvements for cases in which the flow reaches a steady state solution with most RANS modelling approaches. Since the model accounts for the advection of the strains, it is likely that improvements might be brought to regions of flow in which large gradients of strain are apparent, as is the case for streamline curvature or in impinging flows for example. At this point there is no proof of this behaviour and so further testing is necessary in order to provide a conclusive result.

- **Incorporation of the $C_{as}$ model into a hybrid RANS-LES scheme:** Currently, a significant amount of research is focused upon the coupling of URANS schemes with the more exact yet more costly approach of Large Eddy Simulation. Many such techniques have been developed, but one common shortfall remains in the accuracy of the selected URANS modelling, which must be expected to be employed over a considerable proportion of the flow if an economic alternative to a full LES computation is envisaged. Consequently, it would be an interesting extension of this work to examine the potential of the SST-$C_{as}$ model within a hybrid RANS-LES framework. Certainly, in view of the improvements reported in this thesis for the use of the SST-$C_{as}$ model over the standard SST model, there is good reason to anticipate that the use of the SST-$C_{as}$ model would also bring some improvements as a replacement for the standard RANS model commonly used in the SST-DES approach.
Bibliography


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$$f_{VD} = 1 - e^{-y^+ / 26}$$
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\[
\begin{align*}
\sigma_{12} & = \frac{1}{2} \left( \sigma_{11} - \sigma_{22} \right) \\
C_{as} & = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} - \sigma_{33} \right)
\end{align*}
\]
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