A new formulation of the $v^2 - f$ model using elliptic blending and its application to heat transfer prediction

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Outline

✓ Existing versions of the $\overline{v^2} - f$ model and their limits

✓ Use of the elliptic blending with a new version of the $\overline{v^2} - f$, with two main objectives:
  ▶ enhancement of robustness
  ▶ correct prediction of the variables near-wall behaviour

✓ Application to heat transfer in vertical flows

✓ Towards further improvements of prediction accuracy
The $\overline{v^2} - f$ model of (Durbin (1991))

- $EVM + \begin{cases} 
  \text{wall-normal Reynolds stress (transport equation for } \overline{v^2}) \\
  \text{wall-normal pressure-strain correlation (elliptic equation for } f) 
\end{cases}$

- Wall-blocking effect and non-locality of turbulent pressure modelled

\[
\frac{D\overline{v^2}}{Dt} = \begin{bmatrix}
k f \\
- \frac{\overline{v^2}}{k} + \nu \frac{\partial \overline{v^2}}{\partial x_j^2} \\
\end{bmatrix} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu_t}{\sigma_k} \right) \frac{\partial \overline{v^2}}{\partial x_j} \right] + L^2 \Delta \overline{f} - \overline{f} = -f_h
\]

leading order terms

$O(y^4)$

- Leading order terms should balance (to get $\overline{v^2} = O(y^4)$)

- Code friendly model: only implicit near-wall terms $\Rightarrow \delta t$ such that $CFL = 1$

- But here: **Explicit** near-wall source term $f \Rightarrow$ B.C. stiffness $\Rightarrow \delta t \sim T_{Kol}$

$f_w = \frac{-20\nu^2\overline{v^2}}{\varepsilon y^4}$
Previous code friendly versions


\[
\frac{D\bar{v}^2}{Dt} = -6 \frac{\nu^2}{k^2} \varepsilon + \nu \frac{\partial \bar{v}^2}{\partial x_j} + kf + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu_t}{\sigma_k} \right) \frac{\partial \bar{v}^2}{\partial x_j} \right] + O(y^4)
\]

\[
L^2 \Delta \bar{f} - \bar{f} = -f_h - 5 \frac{\nu^2}{k^2} \varepsilon - L^2 \Delta \left( 5 \frac{\nu^2}{k^2} \varepsilon \right)
\]

▷ change of variable \( \bar{f} = f + 5 \frac{\nu^2}{k^2} \varepsilon \)

▷ neglect. term = \( O(f \text{ source terms}) \)

▷ only implicit near-wall source terms

\( \Rightarrow \bar{f}_w = 0 \Rightarrow \text{code-friendly} \)
Previous code friendly versions


\[
\frac{Dv^2}{Dt} = -6 \frac{v^2}{k^2} \epsilon + \nu \frac{\partial v^2}{\partial x_j} + k f + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_k} \frac{\partial v^2}{\partial x_j} \right) + \mathcal{O}(y^4)
\]

\[L^2 \Delta f - f = -f_h - 5 \frac{v^2}{k^2} \epsilon - L^2 \Delta \left( \frac{5}{k^2} \epsilon \right)\]

▷ change of variable \( \bar{f} = f + 5 \frac{v^2}{k^2} \epsilon \)

▷ neglect. term = \( \mathcal{O}(f \text{ source terms}) \)

▷ only implicit near-wall source terms

⇒ \( \bar{f}_w = 0 \) ⇒ code-friendly

✓ Hanjalić et al. (2004), Laurence et al. (2004): \( \varphi = \frac{v^2}{k} \)

\[
\frac{D\varphi}{Dt} = f + \frac{2 \nu}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \nu \frac{\partial \varphi^2}{\partial x_j} - \frac{P}{k} \varphi + \frac{2 \nu_t}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_k} \frac{\partial \varphi}{\partial x_j} \right) + \mathcal{O}(y^2)
\]

\[L^2 \Delta \varphi - \varphi = -f_h\]
Previous code friendly versions


\[ \frac{Dv^2}{Dt} = -6 \frac{v^2}{k^2} \epsilon + \nu \frac{\partial v^2}{\partial x_j^2} \]
\[ L^2 \Delta f - f = -f_h - 5 \frac{v^2}{k^2} \epsilon - L^2 \Delta \left( \frac{5 v^2}{k^2} \epsilon \right) \]

▷ change of variable \( \bar{f} = f + 5 \frac{v^2}{k^2} \epsilon \)

▷ neglect. term = \( \mathcal{O}(f \text{ source terms}) \)

▷ only implicit near-wall source terms

\( \Rightarrow \bar{f}_w = 0 \Rightarrow \text{code-friendly} \)

✓ Hanjalić et al. (2004), Laurence et al. (2004): \( \varphi = \frac{v^2}{k} \)

\[ \frac{D\varphi}{Dt} = f + \frac{2 \nu}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \nu \frac{\partial \varphi^2}{\partial x_j^2} \]
\[ L^2 \Delta f - f = -f_h \]

✓ Hanjalić et al. (2004)

▷ Cross diffusion neglected (\( = \mathcal{O}(\text{diff}_\nu \varphi) \))

▷ Explicit near-wall source term: \( f_w = -10 \nu \frac{\varphi}{y^2} \)

▷ Stiffness still present yet reduced \( \Rightarrow \text{code-friendly} \)
Previous code friendly versions


\[
\begin{aligned}
\frac{D\overline{v}^2}{Dt} &= \begin{cases}
\begin{array}{c}
\text{leading order terms} \quad D\overline{v}^2 \\
\text{change of variable } f = f + 5 \frac{v^2}{k^2} \varepsilon
\end{array}
\end{cases} \\
L^2 \Delta f - \overline{f} &= -f_h - 5 \frac{v^2}{k^2} \varepsilon - L^2 \Delta \left(5 \frac{v^2}{k^2} \varepsilon \right)
\end{aligned}
\]

▷ change of variable \( \overline{f} = f + 5 \frac{v^2}{k^2} \varepsilon \)

▷ neglect. term = \( \mathcal{O}(f \text{ source terms}) \)

▷ only implicit near-wall source terms

⇒ \( f_w = 0 \) ⇒ code-friendly

✓ Hanjalić et al. (2004), Laurence et al. (2004): \( \varphi = \overline{v}^2/k \)

\[
\begin{aligned}
\frac{D\varphi}{Dt} &= \begin{cases}
\begin{array}{c}
\text{change of variable } f = f + 5 \frac{v^2}{k^2} \varepsilon
\end{array}
\end{cases} \\
L^2 \Delta f - \overline{f} &= -f_h - 2 \nu k \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} - 2 \nu \frac{\partial \varphi}{\partial x_j} + 2 \nu \frac{\partial \varphi}{\partial x_j} - L^2 \Delta \left(2 \nu \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \nu \frac{\partial \varphi}{\partial x_j} \right)
\end{aligned}
\]

✓ Laurence et al. (2004)

▷ change of variable \( \overline{f} = f + \frac{2 \nu}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \nu \frac{\partial \varphi}{\partial x_j} \)

▷ Neglected term smaller than other source terms

▷ No more leading-order source terms (not even \( \nu \Delta \varphi \)) ⇒ code-friendly ??
The elliptic blending with an EVM

✓ Elliptic blending introduced in Manceau & Hanjalić (2002) and Manceau (2004) in a Reynolds Stress Model

\[ L^2 \Delta \alpha - \alpha = -1 \quad \Rightarrow \quad f = (1 - \alpha^p) f_w + \alpha^p f_h \]

\[
\frac{D\varphi}{Dt} = \left( -\varphi \varepsilon \right) (1 - \alpha^p) + \nu \frac{\partial \varphi^2}{\partial x_j^2} - \alpha^p f_h - P \frac{\varphi}{k} + 2 \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu_t}{\sigma_k} \right) \frac{\partial \varphi}{\partial x_j} \right]
\]

Leading order terms

\[ O(y^2) \]

✓ The \( \varphi - \alpha \) model combines:

▷ **Implicit** near-wall balance treatment \( \Rightarrow \alpha_w = 0 \Rightarrow \text{code-friendly} \)

▷ No change of variable \( \Rightarrow \text{No terms neglected} \)

▷ Use of the reduced variable \( \varphi \Rightarrow \text{rather than} \ \overline{v^2} \Rightarrow \text{enhanced robustness compared with LDM} \)

▷ Cross diffusive term properly computed
Comparison of the different versions
(Channel flow, $Re^* = 395$)

✓ Unlike $\phi - \bar{f}$, correct near-wall asymptotic behaviour for $\bar{v^2}$ hence $\nu_t$

✓ No over-prediction of $f$ (hence $\bar{v^2}$) in the core region, unlike Lien & Durbin (1996)
Application to turbulence impairment by buoyancy aiding flow)

Forced, mixed, natural convection in a heated pipe (You et al. (2003))

✓ $Re^* = 180$
✓ $Gr/Re^2 = 0.063$ (forced/mixed convection), $Gr/Re^2 = 0.087$ (re-laminarization), $Gr/Re^2 = 0.241$ recovery

Combined natural and forced convection (Kasagi and Nishimura (1997))

✓ Upward flow in a vertical channel
✓ $Re^* = 150, Gr = 9.6 \times 10^5$
✓ Anisotropy enhancement in the buoyancy aiding side
✓ Simple gradient hypothesis for temperature turbulent transport in both cases
Heated pipe: results

Gr/Re^2 = 0.087

Gr/Re^2 = 0.241

Gr/Re^2 = 0.400
Heated vertical channel: results

- DNS
- LDM
- $\varphi - \bar{f}$
- $\varphi - \alpha$
- $k - \omega$ SST

For aiding flow, the plots show the distributions of $U^+$, $k^+$, and $v^+ / k^+$ with $Y^+$ as the independent variable. Opposing flow results are also presented.
Channel flow, $Re^* = 395$, $\nu$/log/defect layer ?

![Graph showing flow characteristics](image-url)
Response to $\varphi - \alpha$ constants (1/2)
Response to $\varphi - \alpha$ constants (1/2)

\begin{align*}
C_{\epsilon 1} &= 1.34 \\
C_{\epsilon 2} &= 1.54 \\
C_{1} &= 0.7 \\
C_{2} &= 2.7 \\
C_{12} &= 1.73 \\
C_{2} &= 2.2
\end{align*}
Conclusions

✓ 4 different versions of the $\overline{v^2} - f$ model revisited $\Rightarrow$ $\varphi - \alpha$ model based on elliptic blending $\Rightarrow$ **numerical stability** improved while respecting known asymptotic states

✓ accurate prediction of **relaminarization** / anisotropy enhancement by aiding buoyancy

✓ Next step: investigation of the $\varepsilon$ equation to:

  ▶ input some ingredients of Launder Sharma model to predict laminar/turbulent transition

  ▶ **extra** information ($\nu \Delta k$, $\frac{\partial \nu \omega}{\partial y}$, $\frac{\partial^2 U}{\partial y^2}$) for **extra** degrees of freedom $\Rightarrow$ a relevant constant tuning
The constants of the $\varphi - \alpha$ model

- $k - \varepsilon$ equations, used with $C_{\varepsilon}^{*} = 1.44(1 + 0.04(1 - \alpha^{p})\sqrt{1/\varphi})$,
  
  $C_{\varepsilon 2} = 1.83$, $\sigma_{k} = 1$

- SSG model: $f_{h} = -\frac{1}{T} \left( C_{1} - 1 - C_{2}' \frac{P}{\varepsilon} \right) \left( \varphi - \frac{2}{3} \right)$, with $C_{1} = 1.7$ and $C_{2} = 1.2$

- $T = \max \left( \frac{k}{\varepsilon}, C_{T} \left( \frac{\nu}{\varepsilon} \right)^{1/2} \right)$, $L = C_{L} \max \left( \frac{k^{3/2}}{\varepsilon}, C_{\eta} \left( \frac{\nu^{3/4}}{\varepsilon^{1/4}} \right)^{1/2} \right)$, $C_{T} = 6$,
  
  $C_{L} = 0.161$, $C_{\eta} = 90$
The constants of the $\varphi - \alpha$ model

✓ $k - \varepsilon$ equations, used with $C_{\varepsilon 1}^* = 1.44(1 + 0.04(1 - \alpha^2)\sqrt{1/\varphi})$, $C_{\varepsilon 2} = 1.83$, $\sigma_k = 1$

✓ SSG model: $f_h = -\frac{1}{T} \left( C_1 - 1 - C'_2 \frac{P}{\varepsilon} \right) \left( \varphi - \frac{2}{3} \right)$, with $C_1 = 1.7$ and $C_2 = 1.2$

✓ $T = \max \left( \frac{k}{\varepsilon}, C_T \left( \frac{\nu}{\varepsilon} \right)^{1/2} \right)$, $L = C_L \max \left( \frac{k^{3/2}}{\varepsilon}, C_\eta \left( \frac{\nu^{3/4}}{\varepsilon^{1/4}} \right)^{1/2} \right)$, $C_T = 6$,
  $C_L = 0.161$, $C_\eta = 90$

✓ $p=2$ (Manceau (2005))

\[
\frac{D \varphi}{Dt} = \alpha^2 f_h - \varphi \frac{\varepsilon}{k} (1 - \alpha^2) + \nu \frac{\partial \varphi^2}{\partial x_j^2} - P \varphi \frac{k}{\sigma_k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu_t}{\sigma_k} \right) \frac{\partial \varphi}{\partial x_j} \right]
\]

leading order terms

\[\mathcal{O}(y^2)\]

(1)
The constants of the $\varphi - \alpha$ model

✓ $k - \varepsilon$ equations, used with $C_{\varepsilon_1} = 1.44(1 + 0.04(1 - \alpha^3))\sqrt{1/\varphi}$, $C_{\varepsilon_2} = 1.83$, $\sigma_k = 1$

✓ SSG model: $f_h = -\frac{1}{T}\left(\frac{C_1 - 1 - C_2'}{\varepsilon}\right)\left(\varphi - \frac{2}{3}\right)$, with $C_1 = 1.7$ and $C_2 = 1.2$

✓ $T = \max\left(\frac{k}{\varepsilon}, C_T\left(\frac{\nu}{\varepsilon}\right)^{1/2}\right)$, $L = C_L \max\left(\frac{k^{3/2}}{\varepsilon}\right)$, $C_T = 6$, $C_L = 0.161$, $C_\eta = 90$

✓ $p=3$

\[ \frac{D\varphi}{Dt} = -\frac{\varphi \varepsilon}{k}(1 - \alpha^3) + \nu \frac{\partial^2 \varphi}{\partial x_j^2} - \alpha^3 f_h - P\varphi - \frac{2 \nu_t}{k} \frac{\partial k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu_t}{\sigma_k} \right) \frac{\partial \varphi}{\partial x_j} \right] \]

leading order terms

$O(y^2)$

(2)
Near wall model for $P_1 + P_2$

✓ It is the term $C_{\varepsilon_1} \frac{P_T}{T} - P_1 - P_2$ which is actually modelled by:

▷ $C_{\varepsilon_1} C_{A1} \frac{P_T}{\varepsilon} P_T$ in Durbin(1993)

▷ $C_{\varepsilon_1} C_{A1} \sqrt{\frac{k}{\nu^2}} \frac{P_T}{T}$ in Parneix(1998)

▷ $C_{\varepsilon_1} C_{A1} \sqrt{\frac{k}{\nu^2}} (1 - \alpha^3) \frac{P_T}{T}$ in Manceau(2004)

▷ The $E$ term in Launder & Sharma(1974)

✓ Their influence do not have the same extent

✓ Addition terms can be added at light of the previous analysis

▷ $C_{\varepsilon_2} f_\varepsilon \nu \frac{\partial^2 k}{\partial y^2} \frac{1}{T}$

▷ $-2\nu \frac{\partial u v}{\partial y} \frac{\partial^2 U}{\partial y^2}$

▷ $0.4\nu \frac{k}{\varepsilon} \frac{\partial v^2}{\partial y} \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2}$
Additionnal terms

\[ P_1 + P_2 - 1.44 \frac{P}{T} \]
\[ 1.44 \times 0.1 \frac{P}{\varepsilon} \frac{P}{T} \]
\[ 1.44 \times 0.045 \frac{\sqrt{k}}{v^2} \frac{P}{T} \]
\[ 1.44 \times 0.045 (1 - \alpha^2) \frac{\sqrt{k}}{v^2} \frac{P}{T} \]
\[ C_{\varepsilon^2} f \nu k_{yy} / T \]
\[ 0.4 \nu k / \varepsilon v^2 U_y U_{yy} \]
\[ -2 \nu \bar{uv}_{yy} \]
 Additionnal terms (terms multiplied by \( y^2 \))

\[
P_1 + P_2 - 1.44P/T \\
1.44*0.1*P/T \\
1.44*0.045*\sqrt{y_2}P/T \\
1.44*0.045*(1-\alpha^2)\sqrt{y_2}P/T \\
C_{y^2}f_{\varepsilon}y\frac{k_{yy}}{T} \\
0.4*\frac{k}{\varepsilon}U_y U_{yy} \\
-2\frac{\nu}{\varepsilon}U_y U_{yy}
\]