FLUID MECHANICS

Lecture 7 Exact solutions
Scope of Lecture

• To present solutions for a few representative laminar boundary layers where the boundary conditions enable exact analytical solutions to be obtained.
Solving the boundary-layer equations

- The boundary layer equations

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \]

\[ \rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \mu \frac{\partial^2 U}{\partial y^2} \]

- Solution strategies:
  - Similarity solutions: assuming “self-similar” velocity profiles. The solutions are exact but such exact results can only be obtained in a limited number of cases.
  - Approximate solutions: e.g. using momentum integral equ’n with assumed velocity profile (e.g. 2nd Yr Fluids)
  - Numerical solutions: finite-volume; finite element,…

- CFD adopts numerical solutions; similarity solutions useful for checking accuracy.
SIMILARITY SOLUTIONS

- Incompressible, isothermal laminar boundary layer over a flat plate at zero incidence (Blasius (1908))

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \nu \frac{\partial^2 U}{\partial y^2}
\]

\[
U_{\infty} = \text{const.} \quad \frac{dP_e}{dx} = 0
\]

\[
U = V = 0 \text{ at } y = 0;
\]

\[
U \rightarrow U_{\infty} \text{ as } y \rightarrow \infty
\]

- Partial differential equations for \(U\) and \(V\) with \(x\) and \(y\) as independent variables.

- For a similarity solution to exist, we must be able to express the differential equations AND the boundary conditions in terms of a single dependent variable and a single independent variable.
THE SIMILARITY VARIABLE

- Condition of similarity
  \[ \frac{U}{U_\infty} = F(\eta) \]

- \( \eta \) is likely to be a function of \( x \) and \( y \)

- Strategy: to replace \( x \) and \( y \) by \( \eta \)

- Similarity variable
  \[ \eta = \frac{y}{\delta} \]

- How to find \( \delta \) before solving the equation?
THE SIMILARITY VARIABLE

• To find the order of magnitude of $\delta$

\[
[\text{Re}_L] = O \left( \frac{1}{\delta^2} \right) \quad \delta \to \frac{\delta}{L} \quad [\text{Re}_L] = O \left( \frac{L^2}{\delta^2} \right) \quad L \to x \quad [\text{Re}_x] = O \left( \frac{x^2}{\delta^2} \right)
\]

\[
\rightarrow \quad [\delta] = O \left[ \frac{x}{\sqrt{\text{Re}_x}} \right] \quad \text{Re}_x = \frac{U_\infty x}{\nu} \quad \rightarrow \quad [\delta] = O \left[ \frac{\sqrt{\nu x}}{U_\infty} \right]
\]

• Similarity variable:

\[
\eta = \frac{y}{O[\delta]} = \frac{y}{\sqrt{\nu x / U_\infty}}
\]
NON-DIMENSIONAL STREAM FUNCTION

• For incompressible 2D flows stream function $\psi$ exists.

\[
U = \frac{\partial \psi}{\partial y} \quad V = -\frac{\partial \psi}{\partial x}
\]

• Strategy: To replace $U$ and $V$ with $\psi$

• Define a non-dimensional stream function $f$ such that $f$ is a function of $\eta$ only.
• Find the order of magnitude of stream function $\psi$

$$U = \frac{\partial \psi}{\partial y}$$

$$[\psi] = O[U_\infty \delta] \quad [\delta] = O\left[\sqrt{\frac{v x}{U_\infty}}\right] \quad [\psi] = O[\sqrt{v U_\infty} x]$$

• Non-dimensional stream function

$$f = \frac{\psi}{\sqrt{v U_\infty} x} \quad \Rightarrow \quad \psi = \sqrt{v U_\infty} x f$$
SIMILARITY SOLUTIONS

• Convert the boundary layer equations

\[
\begin{align*}
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \\
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= \nu \frac{\partial^2 U}{\partial y^2}
\end{align*}
\]

Automatically satisfied by \( \psi \)

\[
f \frac{d^2 f}{d \eta^2} + 2 \frac{d^3 f}{d \eta^3} = 0
\]

or

\[
f f'' + 2 f''' = 0
\]

Substituting

\[
U = \frac{\partial \psi}{\partial y} \quad V = -\frac{\partial \psi}{\partial x}
\]

\[
\psi = \sqrt{\nu U_\infty x f}
\]

so as to replace \( U, V \) by \( f \)

Let \( \eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_\infty}} \)

Replace \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y^2} \)

by derivatives WRT \( \eta \)
Some details of analysis

• Thus with: \( \psi \equiv \sqrt{\nu x U_{\infty}} f(\eta) \quad \eta \equiv y \frac{\sqrt{U_{\infty}}}{\nu x} \)

• we find: \( U = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = \sqrt{\nu x U_{\infty}} f'(\eta) \sqrt{\frac{U_{\infty}}{\nu x}} = U_{\infty} f' \)

\[
V = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{\nu U_{\infty}}{x}} (\eta f' - f) ; \quad \frac{\partial U}{\partial y} = \frac{\partial U}{\partial \eta} \frac{\partial \eta}{\partial y} = U_{\infty} f'' \sqrt{\frac{U_{\infty}}{\nu x}}
\]

• So finally: \( f f'' + 2 f''' = 0 \)

• with b.c.’s \( \eta = 0: f = f' = 0; \eta \to \infty: f' \to 1 \)
SIMILARITY SOLUTIONS

• Substituting into the boundary layer equation

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \nu \frac{\partial^2 U}{\partial y^2}
\]

\[
f \frac{d^2 f}{d \eta^2} + 2 \frac{d^3 f}{d \eta^3} = 0
\]

or

\[
f f'' + 2 f''' = 0
\]

3\textsuperscript{rd} order ordinary differential equation

\[
\begin{cases}
\text{at wall } \psi = 0, \text{ hence } f = \frac{\psi}{\sqrt{\nu U_{\infty} x}} = 0 \\
\text{at wall } U = 0, \text{ hence } f' = \frac{U}{U_{\infty}} = 0 \\
\text{at } y = \infty, \ U = U_{\infty}, \text{ hence } f' = \frac{U}{U_{\infty}} = 1
\end{cases}
\]
The tabulated Blasius velocity profile can be found in many textbooks
VELOCITY PROFILES

\[ \frac{U}{U_\infty} = f'(\eta) \]

\[ \delta \left( \frac{U_\infty}{v_x} \right) \]

\[ \eta = y \sqrt{\frac{U_\infty}{v_x}} \]

\[ \frac{V}{U_\infty} \sqrt{\frac{U_\infty x}{v}} \]

\[ \eta = y \sqrt{\frac{U_\infty}{v_x}} \]

\[ V \text{ is much smaller than } U. \]
DISPLACEMENT AND MOMENTUN THICKNESS

- Boundary layer thickness
  \[ \delta = \frac{5x}{\sqrt{\text{Re}_x}} \]

- Displacement thickness
  \[ \delta^* = \int_0^\infty \left(1 - \frac{U}{U_\infty}\right) dy = \frac{\partial y}{\partial \eta} \int_0^\infty \left(1 - f'\right) d\eta = 1.7208 \sqrt{\frac{\nu x}{U_\infty}} \]
  \[ \delta^* = \frac{1.7208x}{\sqrt{\text{Re}_x}} \]

- Momentum thickness
  \[ \theta = \int_0^\infty \frac{U}{U_\infty} \left(1 - \frac{U}{U_\infty}\right) dy = \frac{\partial y}{\partial \eta} \int_0^\infty f' \left(1 - f'\right) d\eta = 0.664 \sqrt{\frac{\nu x}{U_\infty}} \]
  \[ \theta = \frac{0.664x}{\sqrt{\text{Re}_x}} \]

- \( \delta, \delta^*, \theta \) all grow with \( x^{1/2} \)
  - \( \delta^*/\delta = 0.344, \theta/\delta = 0.133 \)
DISPLACEMENT AND MOMENTUN THICKNESS

- Typical distribution of $\delta$, $\delta^*$ and $\theta$

$U_\infty = 10 \text{ m/s}, \quad v = 17 \times 10^{-6} \text{ m}^2/\text{s}$

![Graph showing the variation of $\delta$, $\delta^*$, and $\theta$ with $x$]
Other Useful Results

• Shape factor:

\[ \delta^* = \frac{1.7208x}{\sqrt{\text{Re}_x}}, \quad \theta = \frac{0.664x}{\sqrt{\text{Re}_x}} \quad \Rightarrow \quad H = \frac{\delta^*}{\theta} = 2.59 \]

• Wall shear stress:

\[ C_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} = \frac{\mu}{\frac{1}{2} \rho U^2} \left( \frac{\partial U}{\partial y} \right)_w = 2 f''(0) \sqrt{\frac{\nu}{U\infty x}} = 0.664 \sqrt{\frac{\nu}{\text{Re}_x}} \]
Short Problem

The boundary layer over a thin aircraft wing can be treated as that over a flat plate. The speed of the aircraft is 100m/s and the chord length of the wing is 0.5m. At an altitude of 4000m, the density of air is 0.819kg/m$^3$ and the kinematic viscosity is $20 \times 10^{-6}$m$^2$/s, Assuming the flow over the wing is 2D and incompressible,

- Calculate the boundary-layer thickness at the trailing edge
- Estimate the surface friction stress at the trailing edge
- Will the boundary layer thickness and friction stress upstream be higher or lower compared to those at the trailing edge?
- What will be the boundary-layer thickness and the surface friction stress at the same chord location if the speed of the aircraft is doubled?
SOLUTIONS

• The Reynolds number at $x=0.5m$:

$$\text{Re}_x = \frac{Ux}{\nu} = \frac{100 \times 0.5}{20 \times 10^{-6}} = 2.5 \times 10^6$$

• The boundary layer thickness:

$$\delta = \frac{5x}{\sqrt{\text{Re}_x}} = \frac{5 \times 0.5}{\sqrt{2.5 \times 10^6}} = 0.0016m$$

• Friction stress at trailing edge of the wing

$$\tau_w = 0.5 \rho u^2 C_f$$

$$= 0.5 \rho u^2 \frac{0.664}{\sqrt{\text{Re}_x}}$$

$$= \frac{0.5 \times 0.819 \times 100^2 \times 0.664}{\sqrt{2.5 \times 10^6}} = 1.72N/m^2$$
Will the boundary layer thickness and friction stress upstream higher or lower compared to those at the trailing edge?

- $\delta$ is smaller upstream as $\delta$ is proportional to $x^{1/2}$
- $\tau_w$ is higher upstream as $\delta$ is thinner.

If the speed of the aircraft is doubled, $Re_x$ will be doubled since $Re_x = \frac{Ux}{\nu}$

- $\delta$ will be smaller as $Re$ increases. $\delta = \frac{5x}{\sqrt{Re_x}}$
- $\tau_w$ will be higher as the increase in $u_\infty$ has a greater effect than the increase in $Re_x$.

$$\tau_w = 0.5 \rho u_\infty^2 \frac{0.664}{\sqrt{Re_x}}$$
Plane stagnation flow

- Flows with pressure gradients can be self-similar ... but it has to be a pressure gradient compatible with self-similarity. See Schlichting and other “advanced” textbooks on fluid mechanics for examples.
- Stagnation flow provides one such example where 
  \( U_e = U_0 x / L \) and \( V_e = -U_0 y / L \) (potential flow)
- Note \( -\frac{1}{\rho} \frac{dP_e}{dx} = U_e \frac{dU_e}{dx} \) by Euler equ’n.
- The boundary layer equation thus becomes:
  \[
  U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{U_0^2}{L} \frac{x}{L} + \nu \frac{\partial^2 U}{\partial y^2}
  \]
- or \( f'''' + fff'' + 1 - f' ^2 = 0 \)
- Here primes denote diff’n wrt \( \eta \equiv y \sqrt{ \frac{U_0}{\nu L} } \)
Stagnation flow results

• Note that the boundary layer has a constant thickness!
• However, the mass within the boundary layer increases continuously since the velocity rises linearly with distance from the stagnation point.
• Unlike the flat-plate boundary layer, the shear stress decreases continuously from the wall.
• For this flow $H = \delta*/\theta = 2.21$ i.e. less than for the zero-pressure-gradient boundary layer.
Asymptotic Suction Flow

- Sometimes it may be desirable to withdraw fluid through the wall

- If the suction is uniform a point is reached where the boundary layer no longer grows with distance downstream and no further change with $x$ occurs.

- Thus the continuity equation is just $\partial V/\partial y = 0$, i.e. $V = -V_w$

- The $x$-momentum equation becomes:

$$-V_w \frac{dU}{dy} = \nu \frac{d^2 U}{dy^2}$$

- ...which is readily integrated to give

$$\frac{U}{U_\infty} = 1 - \exp\left(-\frac{y V_w}{\nu}\right)$$

- A question for you: What is the skin friction coefficient?