Lengthscale-Determining Equations

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Introduction

- Most of the modelling discussed so far has been based on the use of an \( \varepsilon \) equation to provide a turbulence lengthscale.
- However, as already seen, this is not the only possible route: one can solve for one or more other quantities instead.
- Even if one retains an \( \varepsilon \) equation, it is worth considering whether the modelled source/sink terms met so far could be improved upon.
- In non-equilibrium flows it might be questionable whether the use of single velocity and length scales is sufficient.
- This lecture thus aims to:
  - Examine some weaknesses and alternative forms of the \( \varepsilon \) equation.
  - Explore the use of other variables in a second transport equation.
  - Briefly consider the use of multi-scale models.

Time & Length Scales in Turbulence Modelling

- Most turbulence models require at least:
  - A velocity and a length scale (or time and length scales)
  - Some tensorial information regarding the turbulence.
- For example, a linear \( k-\varepsilon \) model represents the stress tensor via the stress-strain relation, and models the turbulent viscosity from a combination of velocity (\( k^{1/2} \)) and length (\( k^{3/2}/\varepsilon \)) scales.
  \[
  \overline{u_iu_j} = -v_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + (2/3)k\delta_{ij} \quad (1)
  \]
  \[
  v_t = c_\mu k^{2}/\varepsilon \quad (2)
  \]
- A stress transport model obtains the tensorial information from solving transport equations for \( \overline{u_iu_j} \), but uses time and length scales in modelling the processes appearing in these equations.
  \[
  \frac{D\overline{u_iu_j}}{Dt} = P_{ij} + \phi_{ij} - \varepsilon_{ij} + d_{ij} \quad (3)
  \]

- The turbulent kinetic energy (obtained either directly from its transport equation or from summing the three normal stresses, \( \overline{u_i^2} / 2 \)) is generally used to provide a velocity scale.
- In one-equation models the lengthscale is prescribed. In more advanced models it is obtained by solving a second transport equation.
- The most widely-used choice is to solve for the turbulent kinetic energy dissipation rate, \( \varepsilon \), but other alternatives have also been proposed.
- Before considering some of these alternatives, we first re-examine the modelling of the \( \varepsilon \) equation.
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An advanced form of stress-transport model developed at UMI ST made two-equation models have usually been tuned to return the same A widely-used (high Reynolds number) form of the log-law in an equilibrium zero-pressure-gradient boundary layer. In fully-turbulent near-wall flows this is

Some models have taken the coefficients $c_{\varepsilon 1}$ and/or $c_{\varepsilon 2}$ to not be simply constants.

For example, the RNG $k$-$\varepsilon$ model makes $c_{\varepsilon 1}$ sensitive to the non-dimensional strain rate, taking

$$c_{\varepsilon 1} = 1.42 - \frac{c_{\mu} \eta^2 (1 - \eta/\eta_0)}{1 + 0.012 \eta} \quad c_{\varepsilon 2} = 1.68$$

with $c_{\mu} = 0.0845$, $\eta_0 = 4.38$ and $\eta = Sk/\varepsilon$ where $S = (S_i S_j / 2)^{1/2}$ and $S_i = \partial U_i / \partial x_i + \partial U_j / \partial x_j$.

An advanced form of stress-transport model developed at UMIST made $c_{\varepsilon 2}$ sensitive to local turbulence structure by taking it to be a function of stress anisotropy invariants:

$$c_{\varepsilon 1} = 1.44 \quad c_{\varepsilon 2} = \frac{1.92}{1 + 0.7 A_2^{1/2}}$$

where

$$A_2 = a_{ij} a_{ij} \quad A_0 = a_{ij} a_{jk} a_{ki} \quad A = 1 - (9/8) (A_2 - A_0)$$

$$a_{ij} = \bar{u}_i \bar{u}_j - (2/3) \delta_{ij}$$

Predicted Lengthscales with the $\varepsilon$ Equation

Mixing-length and one-equation models rely on a prescribed lengthscale.

In fully-turbulent near-wall flows this is usually taken to increase linearly with distance from the wall – to agree with experimental data in equilibrium boundary layers.

Two-equation models have usually been tuned to return the same lengthscale in simple boundary layers, but are not constrained to do so in other flows.

Whilst this is a desirable feature, very large departures of the lengthscale from its equilibrium value can cause problems.

Two examples are predictions of flow and heat-transfer near reattachment and impingement points.
Sudden Pipe Expansion

- Computations with low-Re $k$-$\varepsilon$ model.
- The large lengthscale implies a high near-wall turbulent viscosity, leading to high heat-transfer rates.

Impinging Jet

- Computed with low-Re $k$-$\varepsilon$ scheme.
- Predicted lengthscales near the stagnation point are much greater than in an equilibrium boundary layer.

Lengthscale Corrections

- To overcome the above weaknesses, a “lengthscale correction” is often added to $\varepsilon$ equations.
- This is generally an additional source term, designed to prevent the model from returning excessively large lengthscales.
- A simple form proposed by Yap (1987) is

$$S_\varepsilon = c_w (\varepsilon^2 / k) \max \left( \left( l / l_e - 1 \right) (l / l_e)^2, 0 \right)$$  \tag{6}$$

where $l = k^{3/2} / \varepsilon$ and $l_e = c_l y$ with $y$ the wall-normal distance and $c_w$ and $c_l$ are constants (typically 0.83 and 2.55).

- One drawback with this is that it does require one to define the wall distance – which may be difficult in complex geometries.
- Hanjalić (1996) proposed a form which compared the lengthscale gradient in the wall-normal direction to $c_l$ instead.

- Iacovides & Raisee (1997) extended this to use the resultant lengthscale gradient, making the correction free of wall-geometry details.
- They proposed

$$Y_{dc} = c_w \frac{\varepsilon^2}{k} \max \left( F(F + 1)^2, 0 \right)$$

where

$$F = \left( \left( \partial l / \partial x_j \right) \left( \partial l / \partial x_l \right) \right)^{1/2} - dl_{ed} dy / c_l$$

$dl_{ed} dy$ is the ‘equilibrium lengthscale gradient’, from differentiating the Wolfshtein 1-equation formulation:

$$dl_{ed} dy = c_l \left[ 1 - \exp(-B_e \tilde{R}_l) \right] + B_e c_l \tilde{R}_l \exp(-B_e \tilde{R}_l)$$

with $c_l = 2.55$, $B_e = 0.1069$, $c_w = 0.83$ and $\tilde{R}_l = k^2 / (\varepsilon \nu)$.

Heat Transfer in Ribbed Pipe:

- - - with Yap; —— with $Y_{dc}$
Alternative Lengthscale-Determining Equations

- Although it is the most widely-used second variable, $\varepsilon$ is not the only possible choice.
- Alternatives that have been proposed include:
  - $\omega \equiv \varepsilon / k$
  - $\tau \equiv k / \varepsilon$
- It is possible to formally convert a modelled $\varepsilon$ equation into an exact equivalent for $\tau$ or $\omega$.
- For example, an equivalent $\omega$ equation can be obtained by writing
  $$\frac{D\omega}{Dt} = \frac{D}{Dt}(\varepsilon/k) = \frac{1}{k} \frac{Dk}{Dt} - \frac{\varepsilon}{k^2} \frac{D\varepsilon}{Dt}$$
  and then substituting the source and diffusion terms from the $k$ and modelled $\varepsilon$ equations into equation (7).
- However, a model produced in this fashion would be expected to show exactly the same characteristics as the original modelled $\varepsilon$ equation.

Wilcoxon’s $k$-$\omega$ Models

- Wilcoxon’s (1988) $k$-$\omega$ model took the form
  $$\frac{Dk}{Dt} = P_k - \omega k + \frac{\partial}{\partial x_j} \left[ (v + v_t/\sigma_k) \frac{\partial k}{\partial x_j} \right]$$
  \hspace{1cm} (8)
  $$\frac{D\omega}{Dt} = c_{\omega 1} \omega P_k - c_{\omega 2} \omega^2 + \frac{\partial}{\partial x_j} \left[ (v + v_t/\sigma_\omega) \frac{\partial \omega}{\partial x_j} \right]$$
  \hspace{1cm} (9)
  $$v_t = c_\mu k / \omega$$
  \hspace{1cm} (10)
  with $c_{\omega 1} = 5/9$, $c_{\omega 2} = 5/6$, $c_\mu = 0.09$, $\sigma_k = \sigma_\omega = 2$.
- Although this form did not include any specific “near-wall” terms, a subsequent proposal (Wilcoxon, 1991) did employ a number of $R_t$-dependent functions in order to improve its near-wall behaviour.
- The model was reported to perform better than $\varepsilon$ based schemes in near-wall adverse pressure gradient flows.

Sofialides’ Non-Linear $k$-$\omega$ Model

- Sofialides (1993) tested Wilcoxon’s $k$-$\omega$ model in an impinging jet flow.
- Although stagnation heat transfer was overpredicted, the very sharp peak often returned by $\varepsilon$-based schemes without a lengthscale correction (see earlier slide) was not present.
- As will be seen later, a linear EVM can be expected to show many weaknesses in complex flows – whether it employs an $\varepsilon$, $\omega$ or any other lengthscale-governing equation.
- One of these weaknesses is an overprediction of stagnation heat transfer in impinging flows.
Sofialides then developed a non-linear EVM (based on Suga’s 1995 version – see later lecture) with an $\omega$ equation.

Good impinging jet stagnation point heat transfer was obtained without any need for lengthscale-correction terms.

However, the model did not perform so well in other near-wall flows (for example, by-pass transition).

Abe et al (2003) developed a non-linear EVM in which they employed an $\omega$ equation.

This was based on an EASM (see later lecture), although with additional empirical terms to produce high levels of near-wall stress anisotropy.

They proposed two versions of the model: one with an $\omega$ equation and one with an $\epsilon$ equation.

Flow over an array of 2-D hills:

One potential problem with $\omega$ is that it goes to infinity at a wall, which is not a convenient boundary condition!

In practice, it is generally recommended that the value of $\omega$ should be prescribed at the near-wall node as

$$\omega = \frac{6v}{c_{w2}y^2} \quad (11)$$

where $y$ is the distance of the node from the wall.

This result arises from considering the near-wall balance of terms in the modelled $\omega$ equation.

Another problem – particularly in external aerodynamic flows – is that the $\omega$ equation tends to show too much sensitivity to free-stream turbulence levels.

One feature of Menter’s (1993) model is that it was designed to retain the near-wall $\omega$ equation, but remove its sensitivity to free-stream turbulence.

It does this by using a blending function to switch from an $\omega$ equation to what is equivalent to an $\epsilon$ equation far from a wall:

$$\frac{D\omega}{Dt} = c_{w1}\frac{\omega P_k}{k} - c_{w2}\omega^2 + \frac{\partial}{\partial x_j} \left( v + v_1/\sigma_{w1} \right) \frac{\partial \omega}{\partial x_j} + 2\left(1 - F_1\right) \frac{\sigma_{w2}}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}$$

where $F_1$ is a function designed to switch between the two models.

Adverse pressure gradient boundary layer over an airfoil.

However, this is still essentially a linear EVM, so does not perform particularly well in many complex flows (such as impinging ones).
Multi-Scale Models

- As indicated earlier, the $\epsilon$ equation modelling is based on what is happening in predominantly the large-scale eddies.
- There is thus an implied assumption that the turbulence is in (or close to) spectral equilibrium, i.e. the rate of transfer of energy from the smallest scales (dissipation) is equal to the rate at which it is transferred from the large to the intermediate scales.
- This is unlikely to be true in complex flows. In rapidly evolving flows (in either space or time), the turbulence spectrum will not be in equilibrium.
- The idea behind multiscale models is to attempt to account for this spectral non-equilibrium.

The HLS Model

- Hanjalić, Launder & Schiestel (1980) proposed a multiscale model:
  \[
  \frac{Dk_P}{Dt} = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - \epsilon_P + D_{k_P}
  \]
  \[
  \frac{Dk_T}{Dt} = \epsilon_P - \epsilon_T + D_{k_T}
  \]
  \[
  \frac{D\epsilon_P}{Dt} = -C_P \frac{\epsilon_P}{k_P} \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} + C_P' \frac{\epsilon_P}{k_P} \frac{\partial U_i}{\partial x_m} \frac{\partial U_j}{\partial x_j} \epsilon_{lmk} \epsilon_{ijk} - C_P2 \frac{\epsilon_P^2}{k_P} + D_{\epsilon_P}
  \]
  \[
  \frac{D\epsilon_T}{Dt} = C_T1 \frac{\epsilon_P \epsilon_T}{k_T} - C_T2 \frac{\epsilon_T^2}{k_T} + D_{\epsilon_T}
  \]
- The motivation behind the model can be explained in terms of a series of reservoirs:
  - Production scale energy, $k_P$, is transferred to $k_T$ at a rate $\epsilon_P$.
  - Transfer range energy, $k_T$, is dissipated at a rate $\epsilon_T$ ($\equiv \epsilon$).

The Pulsed Jet

- Bremhorst et al (2003) applied the HLS scheme to a pulsed round jet.
- Both $k-\epsilon$ and RSM were also tested, but only the HLS scheme reproduced some crucial features of the flow.
- For example, the half-width variation through a cycle:
- Further investigations showed that the modelling of spectral non-equilibrium was not crucial in this case. Instead, it was the form of generation terms in the $\epsilon_P$ equation that improved the predicted results.
Generation Rates of $\varepsilon$

- The form of $\varepsilon_P$ generation in the HLS model is similar to that tested by Hanjalić & Launder (1980) within a 2-equation framework.
- They proposed a generation term for $\varepsilon$ of the form (in a 2-D flow)
  \[-c_{\varepsilon 1} \frac{U^* U}{k} \frac{\partial U}{\partial y} - c_{\varepsilon 3} \frac{\varepsilon}{k} (U^2 - V^2) \frac{\partial U}{\partial x}\]
  with $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 3} = 4.44$, and approximated the difference $(U^2 - V^2)$ by $U^2 - V^2 = 0.33k$
- Improved predictions were reported in the axisymmetric jet, as well as adverse pressure gradient boundary layers.
- The modification as above is not in a coordinate-invariant form, (but could be formulated as one) and does not perform so well in some other flows.

Plane/Round Jet Anomaly

- Measurements show a turbulent plane jet in stagnant surroundings spreads at a slower rate than the equivalent axisymmetric jet.
- Standard $k-\varepsilon$ models generally overpredict the round jet spreading rate $dY_1/2/\partial x$
- Hanjalić & Launder reported improved predictions with the above model.
- The streamwise velocity decay rate is greater in the round than in the plane jet.
- The normal-strain related term is thus more important in the round jet case, tending to increase $\varepsilon$, decrease $k$, and hence decrease the spreading rate.

References


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