Introduction

- The first part of this lecture aims to recall the linear eddy-viscosity schemes studied in previous courses:
  - To consider some alternative forms of models.
  - To consider implementation in numerical flow solvers.
- In the second part of the lecture, attention will be turned to the near-wall region.
- The aim is to explore how viscous and other near-wall effects are built into linear EVM’s to allow them to be applied across the thin near-wall viscous sub-layer.
- We will examine a number of approaches, with some examples of applications.
- Some of the modelling strategies mentioned or referred to will be met again, in more detail, later.

Linear Eddy-Viscosity Models

The problem is to obtain approximations for the Reynolds stresses that appear in the mean momentum equations:

\[
\frac{DU_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu \frac{\partial U_i}{\partial x_j} - u_i u_j \right)
\]  

(1)

Linear eddy-viscosity models employ a stress-strain relation of the form:

\[
u_t = \frac{2}{3} k \delta_{ij} - v_1 \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)
\]  

(2)

The Reynolds stresses are linearly related to mean strains via the turbulent, or eddy, viscosity.

The turbulent viscosity \(v_1\) is obtained in a different manner, depending on the level of modelling employed.

Mixing-Length Models

- In simple mixing-length models the mixing-length \(l_m\) is prescribed algebraically, and \(v_t\) obtained from

\[
v_t = \rho_m \left| \frac{\partial U_j}{\partial y} \right|
\]  

(3)

or, in a more general formulation:

\[
v_t = \rho_m \sqrt{\left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)^2}
\]  

(4)

The lengthscale \(l_m\) is taken to vary linearly with distance from the wall, but with a damping term to account for near-wall viscous effects.

- A typical formulation with the Van-Driest near-wall damping term is:

\[
l_m = \min(\kappa y D, \alpha \delta) \quad \quad D = 1 - \exp(-y^+/A)
\]
One-Equation Models

- In a typical 1-equation model, a transport equation is solved for the turbulent kinetic energy $k$:

$$\frac{Dk}{Dt} = P_k - \varepsilon + \frac{\partial}{\partial x_j} \left[ (v + v_l/\sigma_k) \frac{\partial k}{\partial x_j} \right]$$ (5)

- The generation rate is given by $P_k = -\bar{u_i} \bar{U}_j \frac{\partial U_i}{\partial x_j}$.

- The dissipation rate $\varepsilon$ is modelled as $\varepsilon = k^{3/2}/l_\varepsilon$.

- The turbulent viscosity is then taken as $\nu_t = c_\mu k^{1/2} l_\mu$ (6)

- Both lengthscales $l_\mu$ and $l_\varepsilon$ are typically prescribed as increasing linearly with distance from the wall – although they may have different viscous damping terms associated with them to account for very near-wall viscous effects.

$\varepsilon$ is not the only variable that can be solved for to determine the lengthscale.

- Wilcox (1988), for example, proposed solving an equation for $\omega (\equiv \varepsilon/k)$:

$$\frac{D\omega}{Dt} = c_{\omega 1} \frac{\omega P_k}{k} - c_{\omega 2} \omega^2 + \frac{\partial}{\partial x_j} \left[ (v + v_l/\sigma_\omega) \frac{\partial \omega}{\partial x_j} \right]$$ (9)

$$\nu_t = c_\mu k/\omega$$ (10)

which is claimed to give better results in near-wall adverse pressure gradient flows.

Some alternative lengthscale equations will be considered in more detail in a later lecture.

Two-Equation Models

- In a 2-equation model a second variable (often $\varepsilon$) is solved for:

$$\frac{D\varepsilon}{Dt} = c_1 \frac{\varepsilon}{k} P_k - c_2 \varepsilon^2 + \frac{\partial}{\partial x_j} \left[ (v + v_l/\sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_j} \right]$$ (7)

- The turbulent viscosity is then modelled as

$$\nu_t = c_\mu k^2/\varepsilon$$ (8)

- The model constants $c_{11}$, $c_{12}$, $c_\mu$, and $\sigma_\varepsilon$ are tuned for simple equilibrium shear flows.

- In the near-wall region, where viscous effects are important, a damping function is introduced into $c_\mu$, and additional source terms are sometimes included in the $\varepsilon$ equation.

- In a two-equation model there is no need to prescribe a lengthscale for the turbulence; it now emerges from the solution ($l = k^{3/2}/\varepsilon$).

Numerical Implementation

Implementation of the Momentum Equations

- Representing the Reynolds stresses using a linear EVM, the momentum equations become

$$\frac{DU_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ v_i \frac{\partial U_j}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[ \nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] - \frac{\partial}{\partial x_i} \left( 2/3 k \right)$$ (11)

- The final term, containing the gradient of $k$, can be absorbed into the pressure term.

- Adopting this treatment, the variable in which the ‘pressure’ is stored is actually representing $P + (2/3)\rho k$.

- The diffusive terms can be written as

$$\frac{\partial}{\partial x_j} \left[ (v + v_l) \frac{\partial U_i}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[ \nu_t \frac{\partial U_i}{\partial x_j} \right]$$ (12)
The first of these diffusion terms appears similar to the laminar form, except with a modified “effective” viscosity, \( \nu_{\text{eff}} = \nu_t + \nu \).

The second term (which is negligible in thin shear flows), must simply be added into the momentum equation as a source term.

From a computational point of view, the above equation can thus be treated almost identically to the case of a laminar flow – except that the ‘viscosity’ is now varying in space, and cannot be taken as a constant.

The linear eddy-viscosity formulation is thus relatively easy to implement, and is generally fairly stable, since the turbulent viscosity related diffusion terms can mostly be treated in an implicit manner.

Zero-Equation Models

In this case, all that need be done, having updated the velocity field, is to re-calculate \( \nu_t \) using the new velocity gradients.

One- and Two-Equation Models

In these models, separate transport equations are solved for \( k \) (and \( \epsilon \)).

These equations are still of the general form

\[
\frac{D\phi}{Dt} = \frac{\partial}{\partial x_j} \left( \Gamma \frac{\partial \phi}{\partial x_j} \right) + S_{\phi}
\]

so convection and diffusion terms can be discretized in the same way as was done when considering the momentum equations.

In a finite volume approach, integrating the \( k \) equation over a cell leads to a discretized equation of the form

\[
a_p k_p = \sum a_i k_i + S_u
\]

where the major contribution to \( S_u \) is typically from the integration of \( P_k - \epsilon \) over the control volume.

Linearizing Source Terms

In the case of the \( k \) and \( \epsilon \) equations, there are often quite significant source terms on the right hand side of the difference equations.

This is generally unhelpful from a numerical point of view, as it leads to slow convergence, and can give rise to instabilities.

In an iterative scheme, some of these terms can be linearized and treated in a manner that improves stability.

The source term \( S_u \) in the \( k \) equation can be split into two parts:

\[
S_u = S'_u + S_p k_p
\]

where \( S'_u \) contains positive contributions to \( S_u \) and \( S_p \) is negative.

\( S_p \) can then be transferred onto the left hand side of the equation:

\[
(a_p - S_p)k_p = \sum a_i k_i + S'_u
\]
This increases the diagonal dominance of the coefficient matrix, thus improving stability and convergence rates.

For example, in a 1-equation model, the dissipation term in the $k$ equation leads to a source term $-(k^{3/2}/k_v)\text{Vol}$. The term is negative, so can be transferred to the left hand side by setting

$$S_p = -(k^{1/2}/k_v)\text{Vol}$$

(18)

In a two-equation model, since $k$ must be positive, its dissipation term can still be transferred to the left hand side by writing

$$S_p = -(\varepsilon_p/k_p)\text{Vol}$$

(19)

The source/sink terms in the $\varepsilon$ equation can be treated in a similar manner.

Inclusion of Viscous Effects

- Viscous, or near-wall, effects can be included in a number of ways:
  - Terms in transport equations explicitly involving molecular viscosity
    - “Exact” terms, such as viscous diffusion.
    - Modelled source terms.
  - “Damping” functions, which act to modify model coefficient values, and which may depend on quantities such as
    - Turbulent Reynolds number, $R_t = k^2/\varepsilon \nu$.
    - Wall distance, $y^+ = yk^{1/2}/\nu$.
  - Terms involving wall-distances are less preferable as they become difficult to define in complex geometries.
  - In general, low-Reynolds-number models should asymptote to the high-Re forms seen earlier in regions far from walls.

Low-Reynolds-Number Eddy-Viscosity Models

- An accurate representation of near-wall regions may be necessary in highly non-equilibrium flows, or in computing heat-transfer problems.
- Later lectures will consider wall function approaches which aim to avoid or simplify the resolution of this region.
- In mixing length models, viscous damping terms were introduced to reduce the lengthscale in the near-wall (viscosity-affected) region.
- Here, we mainly consider low-Reynolds-number two-equation EVM’s, which can be applied across the viscous sublayer, down to the wall.
- When solving transport equations for turbulence quantities, a number of issues need to be considered:
  - Viscous diffusion should be included
  - Model coefficients may need changing in low-Re regions
  - Additional, near-wall, model terms may need to be included
  - Wall boundary conditions

The Turbulent Viscosity

- In a simple shear flow, the linear stress-strain relation leads to
  $$\frac{\tau_{uv}}{k} = -c_{\mu} \left( \frac{k}{\varepsilon} \frac{\partial U}{\partial y} \right) = -c_{\mu} S$$
  where $S$ is the non-dimensional strain rate.
- In the fully turbulent near-wall region of a local equilibrium boundary layer, $S \approx 3.3$ and $|\tau_{uv}/k| \approx 0.3$, giving $c_{\mu} \approx 0.09$.
- Plotting $-\tau_{uv}/(kS)$ across channel and boundary layer flows:

![Graphs showing the turbulent viscosity in channel and boundary layer flows](image-url)
The turbulent viscosity (effectively $c_\mu$) needs to be reduced across the viscous region.

A damping function $f_\mu$ is usually introduced:

$$\nu_t = f_\mu c_\mu \frac{k^2}{\epsilon} \quad \text{giving} \quad \frac{\nu \kappa}{t} = -f_\mu c_\mu S$$

$f_\mu$ is typically a function of turbulent Reynolds number.

For example, the Launder-Sharma model takes

$$f_\mu = \exp \left\{ -3.4 \left( 1 + \frac{R_t}{50} \right)^2 \right\}$$

### Wall-Limiting Behaviour

The wall-limiting behaviour of flow quantities can be explored using Taylor series expansions.

Close to a no-slip wall one can write

$$u = a_1 y + a_2 y^2 + a_3 y^3 + \cdots$$

$$v = b_1 y + b_2 y^2 + b_3 y^3 + \cdots$$

$$w = c_1 y + c_2 y^2 + c_3 y^3 + \cdots$$

where $y$ is distance from the wall, and the $a$'s, $b$'s and $c$'s are functions of $x$, $z$ and time, but not $y$.

Continuity ($\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$) gives $b_1 = 0$.

Hence $u \propto y$, $w \propto y$, but $v \propto y^2$ at the wall.

The near-wall behaviour of the stresses is thus

$$\overline{u^2} \propto y^2 \quad \overline{w^2} \propto y^2 \quad \overline{v^2} \propto y^4 \quad \overline{uv} \propto y^3$$

Since we now have

$$\overline{uv} = -c \mu f_\mu \frac{k^2}{\epsilon} \frac{\partial U}{\partial y}$$

a suitable form of $f_\mu$ can be chosen to give the correct $\overline{uv}$ behaviour at the wall.

As $k \propto y^2$ and $\epsilon$ is $O(1)$ near the wall (see later), the above implies that $f_\mu \propto 1/y$ at the wall.

This explains the upturn in the effective $c_\mu$ seen earlier in the immediate near-wall vicinity DNS data.

Note, however, that $\epsilon$ in $\nu_t$ above is sometimes replaced by a slightly different variable, which will imply a different near-wall behaviour of $f_\mu$.

With a linear EVM, one cannot obtain the correct near-wall (or often, in fact, the outer) behaviour of all the normal stress components.

### The $\epsilon$ Equation

The dissipation rate $\epsilon$ reaches a maximum at the wall.

This non-zero value should balance with the viscous diffusion of $k$.

One can show that at the wall

$$\epsilon_w = 2\nu \left( \frac{\partial k^{1/2}}{\partial y} \right)^2$$

Although this could be used as a boundary condition, it is not a very convenient condition to apply.

Instead of solving for $\epsilon$ itself, many $\epsilon$-based models use the “isotropic dissipation rate”, $\tilde{\epsilon}$, defined by

$$\epsilon = \tilde{\epsilon} + 2\nu \left( \frac{\partial k^{1/2}}{\partial x_j} \right)^2$$
Note that this results in $\tilde{\epsilon} = 0$ at the wall, which is a convenient boundary condition.

Typically, beyond around $y^+ \approx 5$, $\tilde{\epsilon}$ and $\epsilon$ are essentially identical.

Since $\tilde{\epsilon} \propto y^2$ at the wall, if the turbulent viscosity is now defined as

$$\nu_t = c_\mu f_\mu \frac{k^2}{\tilde{\epsilon}}$$

then the damping function $f_\mu$ should be proportional to $y$ at the wall.

Additional near-wall source terms are often also included in the modelled $\epsilon$ (or $\tilde{\epsilon}$) equation.

Selected Low-Reynolds Number $k$-$\epsilon$ Schemes

Written in a general form:

$$\frac{Dk}{Dt} = P_k - (\tilde{\epsilon} + D) + \frac{\partial}{\partial x_j} \left( (\nu + \nu_t/\sigma_k) \frac{\partial k}{\partial x_j} \right)$$

$$\frac{D\tilde{\epsilon}}{Dt} = c_\epsilon P_k \frac{\tilde{\epsilon}}{\epsilon} + \frac{\partial}{\partial x_j} \left( (\nu + \nu_t/\sigma_\epsilon) \frac{\partial \tilde{\epsilon}}{\partial x_j} \right)$$

<table>
<thead>
<tr>
<th>Model</th>
<th>$c_\mu$</th>
<th>$f_\mu$</th>
<th>$\sigma_k$</th>
<th>$\sigma_\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS</td>
<td>0.09</td>
<td>$\exp(-3.4/(1 + R_t/50)^2)$</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>CH</td>
<td>0.09</td>
<td>1 - $\exp(-0.0115y^+)$</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>YS</td>
<td>0.09</td>
<td>$[1 - \exp(-a_1 R_y R_t^2 - a_2 R_y^2 - a_3 R_t^2)]^{1/2}$</td>
<td>1.0</td>
<td>1.3</td>
</tr>
<tr>
<td>MK</td>
<td>0.09</td>
<td>$[1 + 3.45/R_t^{1/2}] [1 - \exp(-y^+/70)]$</td>
<td>1.4</td>
<td>1.3</td>
</tr>
<tr>
<td>SZS</td>
<td>0.096</td>
<td>$[1 + 3.45/R_t^{1/2}] [1 - \exp(-y^+/115)]$</td>
<td>0.75</td>
<td>1.45</td>
</tr>
</tbody>
</table>

Alternative Lengthscale Equations

- Models which solve for quantities other than $\epsilon$ also require low-Reynolds-number modifications in near-wall regions.

- Wilcox's (1991) $k$-$\omega$ model ($\omega \propto \epsilon/k$) takes $\nu_t = c_\mu f_\mu k/\omega$:

  $$\frac{D\omega}{Dt} = \frac{\alpha_0 + R_t/R_\omega}{c_\mu f_\mu (1 + R_t/R_\omega)} \frac{\partial \omega}{\partial x_j}$$

  $$f_\mu = \frac{\alpha_0 + R_t/R_\omega}{c_\mu f_\mu (1 + R_t/R_\omega)}$$

- The wall boundary condition is applied at the first interior node:

  $$\omega_t = \frac{6\nu}{c_{\omega_2} y_t^2}$$

- The $\omega$ formulation does have some advantages, particularly with regard to near-wall lengthscapes (see later lecture).

- However, $\omega \to \infty$ at the wall may not be particularly attractive.
The $k$-$\tau$ model of Speziale et al (1990) ($\tau = k/\varepsilon$) takes:

$$
\frac{D\tau}{Dt} = (1 - c_{\tau 1}) \frac{\tau P_k}{k} + (c_{\tau 2} f_1 f_2 - 1) + \frac{2}{k} (v + v_t/\sigma_t) \frac{\partial k}{\partial x_j} \frac{\partial \tau}{\partial x_j}
$$

$$
- \frac{2}{\varepsilon} (v + v_t/\sigma_t) \frac{\partial \tau}{\partial x_j} \frac{\partial \tau}{\partial x_j} + \frac{\partial}{\partial x_j} \left( (v + v_t/\sigma_t) \frac{\partial \tau}{\partial x_j} \right)
$$

$f_1 = 1 - \frac{2}{9} \exp\{- (R_t/6)^2 \}$

$f_2 = \left[ 1 - \exp(-y^+/4.9) \right]^2$

$v_t = c_\mu f_\mu k \tau$

$f_\mu = \left[ 1 + \frac{3.45}{R_t^{1/2}} \right] \tanh(y^+/70)$

Like the $\varepsilon$-based schemes above, the details differ from model to model, but the principles are generally similar to those already outlined.

Some of these alternatives will be examined in more detail in a later lecture.

Durbin proposed closing the system by solving a further differential equation for $\phi_{22}^*$.

He wrote $\phi_{22}^* = k \nu_{22}$, and solved an “elliptic relaxation” equation for $f_{22}$, of the form

$$
L^2 \frac{\partial^2 \nu_{22}}{\partial x_j^2} - f_{22} = - \Pi_{22}/k - (\nu^2 / k - 2/3)/T
$$

(20)

where the length and time scales $L$ and $T$ are

$L = C_L \max(k^{3/2}/\varepsilon, c_\eta (\nu^3/\varepsilon)^{1/4})$

$T = \max(k/\varepsilon, C_T (\nu/\varepsilon)^{1/2})$

$\Pi_{22}$ is a redistribution model taken from the stress transport model of Launder, Reece & Rodi (1975):

$$
\Pi_{22} = -c_1(\nu^2 - 2k/3)/T + c_2 P_k
$$

Far away from a wall $f_{22}$ should revert to the form of the LRR model, but the differential operator in equation (20) modifies this behaviour near a wall.

Elliptic Relaxation Approaches ($\nu^2$-$f$ Models)

In the forms considered so far we have taken $v_t = c_\mu f_\mu k^2/\varepsilon$, with $f_\mu$ accounting for the reduction in $v_t$ needed near a wall.

Durbin (1991) noted that DNS data showed $c_\mu k \nu^2/\varepsilon$ to give a good fit to the implied turbulent viscosity.

He proposed solving a transport equation for $\nu^2$ – the wall-normal stress component.

In simple shear flow a modelled $\nu^2$ transport equation can be written as

$$
\frac{D\nu^2}{Dt} = \phi_{22}^* - \nu^2 \frac{\varepsilon}{k} + \frac{\partial}{\partial x_j} \left( (v + v_t) \frac{\partial \nu^2}{\partial x_j} \right)
$$

where $\phi_{22}^*$ represents a combination of redistribution and dissipation processes (see later lecture on stress transport models).

Durbin showed good results in plane channel flow.

A number of modified forms of this $\nu^2$-$f$ model approach have been proposed.
Simple Near-Wall Flows

- Near-wall $k$ profiles for several low-Re EVM's (from Sarkar & So, 1997).
- Most models perform reasonably.
- Note some models are tuned to give correct near-wall asymptotes in these flows.

Accelerating Boundary Layers

- Acceleration parameter $K$ defined as
  $$K = \frac{\nu}{U_\infty^2} \frac{dU_\infty}{dx}$$
- Jones & Launder (1972)

Skewed Channel Flow

- Starts with fully developed channel flow.
- Walls are then impulsively moved in direction perpendicular to flow direction.
- DNS data of Howard & Sandham (2000) shows an initial reduction in turbulence energy and shear stress, followed by a recovery as the flow re-aligns to the new direction.
Skewed Channel: Shear Stress at Selected Times

Launder-Sharma $k$-$\varepsilon$

$k$-$\omega$ schemes

Symbols: DNS data

The $E$ source term in the Launder-Sharma model is crucial in obtaining the reduction in peak turbulence levels during the initial skewing.

By-Pass Transition

- Transition position is dependent on free-stream turbulence levels.
- Skin friction
- Velocity profiles
- Reasonable results returned by the Launder-Sharma scheme.

Impinging Flow

- Previous flows have mainly involved obtaining the correct shear stress.
- In more complex flows, EVM's can fail badly.
- Impinging jet heat transfer with Launder-Sharma scheme.

Solid line: Original $k$-$\varepsilon$ form; Broken line: $k$-$\varepsilon$ with lengthscale correction.

We will revisit this problem in a later lecture.

Summary

- We have revisited the general forms adopted for linear EVM's.
- Numerical implementation of linear EVM's is fairly straightforward: the inclusion of turbulent viscosity can help stabilize the system of equations.
- Low-Re-number EVM's generally contain damping terms and near-wall source terms, dependent on viscosity and/or wall distance.
- Examples have been presented of $k$-$\varepsilon$ schemes and some alternative approaches.
- Most perform acceptably in channel or simple boundary layer flow.
- Additional source terms included in some models aid in more complex flows such as accelerating or transitional boundary layers.
- In flows with more complex strains, linear EVM's (high or low Re) may fail.
- All these low-Re schemes require a very fine near-wall grid, which can be expensive in 3-D. Alternative strategies will be examined in a later lecture.
References