Stress Transport Models

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Introduction

- Both the linear and non-linear EVM’s have used *algebraic* relations linking the Reynolds stresses to mean strains.
- This means the stresses respond instantly to changes in the mean strain. It also leads to a direct link between stress anisotropy and mean strains – if mean velocity gradients vanish at some point in the flow, even NLEVM’s return isotropic turbulence at that position.
- We saw earlier that linear and non-linear EVM’s return isotropic turbulence at the centre of a plane channel flow.
- In a plane wall jet the measured stresses are far from isotropic around the velocity maximum.

The Reynolds Stress Transport Equation

- Physically, individual stresses get generated, convected, diffused and dissipated at different rates. We now explore a modelling strategy that takes account of this.
- An exact transport equation can be constructed for $u_i u_j$, by manipulating the equations for $u_i$ and $u_j$. Some elements within these do have to be modelled – but this is now at a more fundamental level than the previous approach of obtaining an effective turbulent viscosity.
- An important aspect of these stress transport equations is that generation terms are represented exactly, so do not require modelling.
- In this section we aim to:
  - Derive the Reynolds stress transport equations.
  - Examine the generation terms and their behaviour in different flows.
  - Consider some basic models for closing the stress equations.
  - Outline numerical solution treatments and some advanced modelling strategies.

\[ \frac{\partial u_i u_j}{\partial t} + U_k \frac{\partial u_i u_j}{\partial x_k} = \cdots (1) \]

- Adding this to the equation for $u_i$ multiplied by $u_j$ (also averaged), results in an equation which begins

\[ \frac{\partial u_i}{\partial t} + U_k \frac{\partial u_i}{\partial x_k} + \nu \frac{\partial^2 u_i}{\partial x_k^2} = \cdots (2) \]

or

\[ \frac{\partial u_i u_j}{\partial t} + U_k \frac{\partial u_i u_j}{\partial x_k} \equiv \frac{\partial u_i u_j}{\partial t} = \cdots (3) \]

- Some of the terms on the right hand side need further manipulation, for example to distinguish diffusion from dissipation, etc.
The stress transport equation then becomes
\[
\frac{\partial \overline{u_i u_j}}{\partial t} = \left( \frac{\overline{u_i u_k}}{\partial x_k} + \frac{\overline{u_j u_k}}{\partial x_k} \right) - 2v \frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \overline{u_i u_j}}{\partial x_k} - \frac{p}{\rho} \frac{\partial ^2 \overline{u_i u_j}}{\partial x_k} \right)
\]

which is often written in shorthand notation
\[
\frac{\partial \overline{u_i u_j}}{\partial t} = P_{ij} - \varepsilon_{ij} + \phi_{ij} + d_{ij}
\]

There are similarities with the turbulent kinetic energy equation:
- \(P_{ij}\) is the generation rate of the turbulent stress by mean strain.
- \(\varepsilon_{ij}\) is the viscous dissipation rate of the stress component.
- \(d_{ij}\) is the diffusion rate of \(\overline{u_i u_j}\) by turbulent and viscous action.

For practical computations models are needed for \(\phi_{ij}\), \(\varepsilon_{ij}\) and \(d_{ij}\). However, the generation terms do not require modelling.

The Production Tensor

- The production terms are exact, in that they only contain Reynolds stresses and mean strains, and therefore do not need further modelling:
  \[
P_{ij} = - \left( \frac{\overline{u_i u_k}}{\partial x_k} + \frac{\overline{u_j u_k}}{\partial x_k} \right)
  \]

- In a simple shear flow, with \(U_1(x_2)\), \(U_2 = U_3 = 0\),
  \[
P_{11} = -2 \overline{u_1 u_2} \frac{\partial U_1}{\partial x_2} \quad P_{22} = P_{33} = 0
  \]

This shows why \(\overline{u_1 u_2}\) is usually of the opposite sign to \(\partial U_1 / \partial x_2\) because its generation term is of the opposite sign.

It also shows why free shear flows spread more rapidly than those near walls: \(u_2^2 / k\) is higher in free flows than in near-wall flows, leading to higher levels of shear stress generation.

- One difference between the turbulent kinetic energy and stress equations is that the \(k\) transport equation has no equivalent of the process \(\phi_{ij}\).

- If we sum the equations for the normal stresses (set \(i = j\)) we produce an equation for twice the turbulent kinetic energy (\(D\overline{u_i u_i}/Dt = 2Dk/Dt\)).

- Contracting \(\phi_{ij}\) we get:
  \[
  \phi_{ij} = \frac{p}{\rho} \left( \frac{\partial \overline{u_i u_j}}{\partial x_i} + \frac{\partial \overline{u_i u_j}}{\partial x_j} \right) = 0 \quad \text{(by continuity)}
  \]

This process (called the “pressure-strain” or “pressure-scrambling” term) thus makes no direct contribution to the level of turbulence energy.

- It does redistribute turbulence energy from one direction to another.

- As a generalization, if the normal stress in one direction is less than in the other directions, it will typically receive energy through \(\phi_{ij}\).

- The process \(\phi_{ij}\) is thus often thought of as tending to return the turbulence towards an isotropic state.

- Of the normal stresses, it is only the streamwise component \(\overline{u_1^2}\) which is directly generated by \(P_{ij}\).

- Examining the terms in the stress transport equations:
  \[
  \frac{D\overline{u_1^2}}{Dt} = -2 \overline{u_1 u_2} \frac{\partial U_1}{\partial x_2} + \phi_{11} - \varepsilon_{11} + d_{11}
  \]

  \[
  \frac{D\overline{u_2^2}}{Dt} = \phi_{22} - \varepsilon_{22} + d_{22}
  \]

  \[
  \frac{D\overline{u_3^2}}{Dt} = \phi_{33} - \varepsilon_{33} + d_{33}
  \]

The dissipation terms, \(\varepsilon_{11}, \varepsilon_{22}\) and \(\varepsilon_{33}\) must be positive, and diffusion is often a relatively small process.

- Since \(\overline{u_2^2}\) and \(\overline{u_3^2}\) are generally non-zero in simple shear flows, the pressure-strain, \(\phi_{ij}\), must act to ‘redistribute’ some of the energy from \(\overline{u_1^2}\) into the \(\overline{u_2^2}\) and \(\overline{u_3^2}\) components.

This explains why the streamwise normal stress is typically larger than the other two components.
In a simple shear flow the way in which turbulence is sustained can thus be represented by a triangular diagram.

- The turbulent shear stress \( \overline{uv} \) is generated by \( \overline{v^2} \) (and velocity gradients).
- The shear stress \( \overline{uv} \) appears in \( P_{11} \), the generation rate of \( \overline{u^2} \).
- The redistribution process, \( \phi_{ij} \), feeds turbulence energy from \( u^2 \) into \( v^2 \) (and \( w^2 \)).

**Generation Rates on Curved Surfaces**

- Even weak surface curvature can have a strong effect on the levels of turbulence. The reasons for this can be seen from the generation terms.
- In addition to the primary shear \( \partial U_1 / \partial x_2 \), there is a shear associated with curvature \( \partial U_2 / \partial x_1 \).

We now have generation rates

\[
P_{12} = -\overline{u_2 v_1} \frac{\partial U_1}{\partial x_2} - \overline{u_1 v_2} \frac{\partial U_2}{\partial x_1} \quad (7)
\]

and

\[
P_{22} = -\overline{u_1 u_2} \frac{\partial U_2}{\partial x_1} \quad (8)
\]

- For a concave wall as shown above \( \partial U_1 / \partial x_2 > 0 \), and \( \partial U_2 / \partial x_1 < 0 \), so both terms in \( P_{12} \) are of the same sign.
- For weak curvature \( \partial U_2 / \partial x_1 \) will be much smaller than \( \partial U_1 / \partial x_2 \).

However, \( \overline{u_2} \) will typically be much larger than \( \overline{u^2} \) (particularly across the viscosity affected sublayer).

- The difference in magnitude of the associated normal stresses means that the second term in \( P_{12} \) is not insignificant in comparison to the first.
- In addition, the generation term in the \( \overline{u^2} \) equation is positive (since \( \overline{u_1 u_2} \) is negative), thus helping to further increase \( P_{12} \).

A linear eddy-viscosity model takes

\[
-\overline{u_1 u_2} = \nu_t \left( \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right) \quad (9)
\]

- In this the turbulent stress is equally sensitive to both primary and secondary strain rates. Since, with weak curvature, the secondary strain is much smaller than the primary, its effect is negligible.
- Likewise, for a boundary layer on a convex surface, the shear stress production rate magnitude is reduced.
Modelling of the Stress Transport Equations

- Although the generation terms are treated exactly, models are still needed for:
  - Dissipation, $\varepsilon_{ij}$; diffusion, $d_{ij}$; and pressure-strain correlation, $\phi_{ij}$.
- There are a number of properties which one might want models to have:
  - The correct tensorial form: to give the same symmetries and contraction properties as the exact processes.
  - Coordinate invariance: models should be independent of the frame of reference – including accelerating or rotating frames.
  - Realizability: models should not predict physically impossible values – such as negative normal stresses.
  - Consistency with physical wall/surface limits.
  - Geometry independence: models should not be dependent on details of the specific geometry being studied.
- Models described in this lecture satisfy the first two constraints.

Dissipation, $\varepsilon_{ij}$

- Dissipative processes arise from the smallest scale eddies:
  \[ \varepsilon_{ij} = 2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \]  
  (10)
- Turbulence energy gets cascaded down from larger eddies to smaller ones.
- If there is a large enough range of scales, one can argue that the smallest eddies will be almost isotropic.
- Hence, at high Reynolds numbers, the dissipative terms are often assumed to be isotropic:
  \[ \varepsilon_{ij} = (2/3)\varepsilon \delta_{ij} \]  
  (11)
- This implies an equal effect on all normal stresses, and none on the shear stress.

Diffusion, $d_{ij}$

- From the exact expression, diffusion can be seen to be due to triple moments, pressure-velocity correlations and viscous effects:
  \[ d_{ij} = -\frac{\partial}{\partial x_k} \left( \nu_t \frac{\partial u_i}{\partial x_k} u_j + p \frac{\delta_{ij}}{\rho} \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \delta_{ij} - \nu \frac{\partial^2 u_i}{\partial x_k^2} \right) \]  
  (12)
- The viscous term is exact, but generally negligible except close to a wall.
- Pressure-diffusion is usually negligible, except very close to a wall or surface.
- The principle of receding influence is often used to argue that higher order moments become less important, and a relatively simple model can be adopted for the triple moments.
- One simple form is a gradient diffusion model, as typically used in the $k$ equation of eddy-viscosity models:
  \[ d_{ij}^* = \frac{\partial}{\partial x_k} \left( \nu_t \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right) \]  
  (13)
A better representation is often provided by the generalized gradient diffusion hypothesis (GGDH) of Daly & Harlow (1970).

This models the flux of some quantity $\phi$ as

$$ u_k \phi \propto -k \frac{\partial \phi}{\partial x_l} \tag{14} $$

Applying this to the triple moments gives

$$ \delta ij = \frac{\partial}{\partial x_k} \left( c_i \frac{k}{\epsilon} u_k u_l \frac{\partial \phi}{\partial x_l} \right) \tag{15} $$

Comparing this to the simpler form of equation (13), we see the GGDH implies a non-isotropic “turbulent viscosity”.

From the above expression one expects to have contributions in $\phi_{ij}$ arising from turbulence-turbulence interactions and from mean strains.

Hence $\phi_{ij}$ is often modelled as

$$ \phi_{ij} = \phi_{ij1} + \phi_{ij2} \tag{18} $$

where $\phi_{ij1}$ depends only on the turbulence, and $\phi_{ij2}$ accounts for mean strain influences.

In a simple shear flow we saw that $\phi_{ij}$ acts to reduce the stress anisotropy – redistributing energy from the streamwise component into the other two normal stresses.

Initially anisotropic turbulence returns towards isotropy once external strains have been removed. This must be due to $\phi_{ij1}$, the “return to isotropy”, or “slow pressure-strain”, term.

A simple linear model for $\phi_{ij1}$ is thus (Rotta 1951):

$$ \phi_{ij1} = -c_1 \epsilon a_{ij} \tag{19} $$

where $a_{ij} = \overline{u_i u_j} / k - (2/3) \delta_{ij}$.

**Pressure-Strain, $\phi_{ij}$**

As already seen, $\phi_{ij}$ is redistributive: it has no direct influence on levels of turbulence energy.

$$ \phi_{ij} = \frac{\rho}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{16} $$

A Poisson equation can be obtained for $p$, and formally solved with the help of a Green’s function, giving an expression for $\phi_{ij}$:

$$ \phi_{ij} = \frac{1}{4\pi} \int_V \left( \frac{\partial^2 \overline{u_i' u_j' u_l'}}{\partial r_i \partial r_j \partial r_l} + \frac{\partial^2 \overline{u_i' u_j' u_l'}}{\partial r_i \partial r_k \partial r_l} \right) \frac{dV}{|r|} $$

$$ - \frac{1}{2\pi} \int_V \left( \frac{\partial \overline{u_i'}}{\partial r_l} \left( \frac{\partial^2 \overline{u_i' u_j' u_l'}}{\partial r_k \partial r_l} + \frac{\partial^2 \overline{u_i' u_j' u_l'}}{\partial r_k \partial r_l} \right) \frac{dV}{|r|} \right. $$

Advanced schemes make direct use of this expression, and attempt to approximate the integrals in it. Here, however, we take a simpler route.

Equation (19) acts to redistribute energy, reducing the anisotropy of the stresses.

The coefficient $c_1$ is typically taken around 1.8.

$\phi_{ij2}$ generally has a similar effect of tending to reduce the stress anisotropy.

A simple (and widely used) model for the “rapid” pressure strain term is

$$ \phi_{ij2} = -c_2 (P_{ij} - (1/3)P_{kk} \delta_{ij}) \tag{20} $$


This acts to redistribute the generation rates, thus reducing anisotropy, and is known as the “Isotropization of Production”, or IP, model.

The coefficient $c_2$ is usually taken around 0.6.
Simple Shear Flow

- In a simple shear $P_{22} = P_{33} = 0$, so the above form for $\phi_{ij}$ gives
  \[ \phi_{112} = -(2/3)c_2 P_{11}, \quad \phi_{222} = (1/3)c_2 P_{11}, \quad \phi_{332} = (1/3)c_2 P_{11}, \quad \phi_{122} = -c_2 P_{12} \]

- $\phi_{ij}$ thus acts as a sink for $u_i^2$ and $u_1 u_2$, but a source for $u_2^2$ and $u_3^2$.

- The modelled stress transport equations in a simple shear thus become
  \[
  \frac{\partial u_i^2}{\partial t} + \frac{\partial \left(u_i u_j u_j\right)}{\partial x_j} = P_{11} - (2/3)c_2 P_{11} - c_1 \varepsilon a_{11} - \varepsilon_{11} + d_{11} \\
  \frac{\partial u_2^2}{\partial t} + \frac{\partial \left(u_1 u_2 u_2\right)}{\partial x_1} = (1/3)c_2 P_{11} - c_1 \varepsilon a_{22} - \varepsilon_{22} + d_{22} \\
  \frac{\partial u_3^2}{\partial t} + \frac{\partial \left(u_1 u_3 u_3\right)}{\partial x_1} = (1/3)c_2 P_{11} - c_1 \varepsilon a_{33} - \varepsilon_{33} + d_{33} \\
  \frac{\partial u_1 u_2}{\partial t} + \frac{\partial \left(u_1 u_2 u_2\right)}{\partial x_1} = P_{12} - c_2 P_{12} - c_1 \varepsilon a_{12} - \varepsilon_{12} + d_{12}
  \]

Accounting for Wall-Reflection Effects

- In simple shear flow with velocity gradient $dU_1/dx_2$, the model described above gives identical values for $u_2^2$ and $u_3^2$.

- This is not what is found experimentally, even in free flows: in near-wall flows even larger anisotropy is expected.

- Pressure fluctuations get reflected from walls or surfaces, and this leads to a damping of the velocity fluctuations normal to the surface.

- Some typical measured stress anisotropy levels are given below:

<table>
<thead>
<tr>
<th></th>
<th>$u_2^2/k$</th>
<th>$u_3^2/k$</th>
<th>$u_1 u_2/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free shear flow</td>
<td>0.95</td>
<td>0.47</td>
<td>0.55</td>
</tr>
<tr>
<td>Near-wall flow</td>
<td>1.20</td>
<td>0.25</td>
<td>0.55</td>
</tr>
</tbody>
</table>

- A widely-used wall correction to $\phi_{ij}$ (Shir, 1973) can be written:
  \[ \phi_{ij}^w = c_{1w} \frac{\varepsilon}{K} \left(\frac{u_i u_j n_j n_k}{\delta_{jk}} - (3/2) \frac{u_i u_j n_j n_k}{\delta_{jk}} \right) f_y \] (22)

where $c_{1w}$ is typically taken as 0.5.

- For a single plane wall with $n_1 = n_3 = 0$, $n_2 = 1$ this gives:
  \[ \phi_{111}^w = \phi_{331}^w = c_{1w} \frac{\varepsilon}{K} u_2^2 f_y \quad \phi_{221}^w = -2 c_{1w} \frac{\varepsilon}{K} u_2^2 f_y \]
  \[ \phi_{121}^w = -(3/2) c_{1w} \frac{\varepsilon}{K} u_1 u_2 f_y \]

so the model does act to impede the transfer of energy into $u_2^2$ as the wall is approached.

- Corrections are usually also applied to $\phi_{ij}$. The Gibson-Lauder (1978) model for $\phi_{ij}^w$ can be written:
  \[ \phi_{ij}^w = c_{2w} \left(\frac{\varepsilon}{K} u_i n_j n_k - (3/2) \phi_{ij} n_j n_k - (3/2) \phi_{jk} n_i n_k \right) f_y \] (23)

with $c_{2w}$ usually taken as 0.3.

- One of the effects of a wall (or surface) is to impede the energy transfer into the stress component normal to the wall.

- This also leads to a reduction of the shear stress.

- This wall damping effect is often modelled via wall-correction terms (sometimes called wall-reflection terms), employing the turbulence lengthscale and wall-distance to determine the strength of corrections.

- The function $f_y$ defined by
  \[ f_y = \frac{k^{3/2}/\varepsilon}{2.5 x_p n_p} \] (21)

  with $\mathbf{n}$ the unit vector normal to the wall, takes a value close to unity near the wall, but decreases as one moves further away.

- Note, however, that this can be difficult to apply consistently when considering complicated geometries.
This acts to oppose the $\phi_{ij2}$ redistribution process.

For a simple shear flow with $n = (0, 1, 0)$ we have

$\phi_{112} = - (2/3) c_2 P_{11}$  
$\phi_{222} = (1/3) c_2 P_{11}$  
$\phi_{333} = (1/3) c_2 P_{11}$  
$\phi_{122} = -c_2 P_{12}$

The above wall-reflection term thus gives

$\phi_{112}^w = \phi_{332}^w = c_2 w \phi_{222} f_y = (1/3) c_2 c_2 w P_{11} f_y$
$\phi_{222}^w = -2 c_2 w \phi_{222} f_y = -(2/3) c_2 c_2 w P_{11} f_y$
$\phi_{122}^w = -(3/2) c_2 w \phi_{122} f_y = (3/2) c_2 c_2 w P_{12} f_y$

The term thus acts to reduce the wall-normal stress $u_u u_v$, and increase the streamwise and transverse ones.

In a plane impinging flow as shown, we have

$P_{11} = -2 \overline{u^2} \frac{\partial U}{\partial x}$
$P_{22} = -2 \overline{v^2} \frac{\partial V}{\partial y} = 2 \overline{v^2} \frac{\partial U}{\partial x}$

$P_{22}$, corresponding to the stress component normal to the wall, is positive, whilst $P_{11}$ is negative.

The IP rapid pressure-strain model now gives

$\phi_{112} = (2/3) c_2 (\overline{u^2} + \overline{v^2}) \frac{\partial U}{\partial x} > 0$  
$\phi_{222} = -(2/3) c_2 (\overline{u^2} + \overline{v^2}) \frac{\partial U}{\partial x} < 0$

The Gibson-Launder wall-reflection terms are then

$\phi_{112}^w = c_2 w \phi_{222} f_y < 0$  
$\phi_{222}^w = -2 c_2 w \phi_{222} f_y > 0$

The above form of wall-reflection thus has the effect of increasing the stress component normal to the wall.

Other forms of wall-reflection terms have been proposed.

For example, the form of Craft & Launder (1992) is designed to have the desired effect (reducing the wall-normal stress) in both parallel and impinging flows:

$$\phi_{ij2}^w = -0.08 \frac{\partial U_j}{\partial x_k} U_j \delta_{ij} n_i n_k - 3 n_i n_j f_y$$
$$-0.1 k a_{in} \left( \frac{\partial U_k}{\partial x_m} n_k n_i \delta_{ij} - (3/2) \frac{\partial U_k}{\partial x_m} n_k n_i n_j - (3/2) \frac{\partial U_k}{\partial x_m} n_i \right) f_y$$
$$+ 0.4 k \frac{\partial U_k}{\partial x_l} n_k n_i \left( n_i n_j - 1/3 n_i n_j \delta_{ij} \right) f_y$$
When considering the implementation of the momentum equation
\[
\frac{DU_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_i} \left( \nu \frac{\partial U_j}{\partial x_j} \right) - \frac{\partial}{\partial x_j} (\overline{u_i u_j})
\]  
(24)

using eddy-viscosity based models, the aim was to minimize the size of the source term arising from the Reynolds stress gradients.

With a NLEVM, the modelled stresses can be split into linear and non-linear parts.

The linear terms can be treated implicitly, whilst the remaining non-linear terms, defined by
\[
\overline{u_i u_j} = u_i u_j - (2/3)k \delta_{ij} - \nu_t S_{ij}
\]  
(25)

are put into the source term as
\[
-\frac{\partial}{\partial x_j} (\overline{u_i u_j})
\]

Such a decomposition leads to increased diagonal terms in the coefficient matrix, which aids stability.

Suitable expressions for the apparent viscosities can be obtained from examining the transport equations for the stresses.

Using the linear model introduced earlier (without wall-reflection):
\[
\frac{D \overline{u^2}}{Dt} = P_{11} + \phi_{11} - \varepsilon_{11} + d_{11}
\]
\[
= -2 \overline{u^2} \frac{\partial U}{\partial x} - 2 \overline{uv} \frac{\partial U}{\partial y} - c_1 \varepsilon (\overline{u^2} / k - 2/3) - c_2 (P_{11} - P_{kk} / 3) - (2/3) \varepsilon
\]

Neglecting convection and diffusion terms this can be re-arranged into an expression of the form
\[
\overline{u^2} = -\mu_{11} \frac{\partial U}{\partial x} + \text{other terms}
\]

where \( \mu_{11} = \left[ \frac{2 - (4/3)c_2}{c_1} \right] \frac{k \overline{u^2}}{\varepsilon} \)

A similar expression can be obtained when wall-reflection terms are included:
\[
\mu_{11} = \left[ \frac{2 - (4/3)c_2 + (2/3)c_2 c_2 w(4f_x + f_y)}{c_1 + 2 c_1 w f_x} \right] \frac{k \overline{u^2}}{\varepsilon}
\]  
(26)

Manipulating the transport equation for \( \overline{uv} \) gives a corresponding expression for \( \mu_{12} \):
\[
\mu_{12} = \left[ \frac{1 - c_2 + 3/2 c_2 c_2 w (f_x + f_y)}{c_1 + 3/2 c_1 w (f_x + f_y)} \right] \frac{k \overline{uv}}{\varepsilon}
\]  
(27)

The other velocity components can also be treated in a similar manner.

In a stress transport scheme, there is no such natural decomposition of the stresses.

Simply including the entire gradient \( -\partial \overline{u_i u_j} / \partial x_j \) in the source term of the momentum equation will often result in instabilities.

To improve the stability of the numerical scheme when using a stress transport model one can introduce the idea of an ‘apparent viscosity’.

Consider the \( U \) momentum equation in 2-D:
\[
\frac{DU}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} \left( \mu \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial U}{\partial y} \right) - \frac{\partial}{\partial x} (\overline{u^2}) - \frac{\partial}{\partial y} (\overline{uv})
\]

The idea is to express the stresses \( \overline{u^2} \) and \( \overline{uv} \) in the form
\[
\overline{u^2} = \overline{u^2} - \mu_{11} \frac{\partial U}{\partial x} \quad \overline{uv} = \overline{uv} - \mu_{12} \frac{\partial U}{\partial y}
\]
The above slides have described (relatively) simple modelling ideas that can be used to close the stress transport equations.

These were based on local isotropy (for the dissipation) and return to isotropy for the pressure-strain correlation.

“Wall-reflection” terms were added to $\phi_{ij}$ to oppose the return to isotropy process near a wall, thus damping the wall-normal stress component.

The most crucial process to be modelled is often the pressure-strain correlation $\phi_{ij}$. More advanced models of this process often adopt forms that are non-linear in the stresses.

Invariants of the stress anisotropy tensor (e.g., $A_2 = a_{ij}a_{ij}$ and $A_3 = a_{ij}a_{jk}a_{ki}$) may also be used in models.

The above scalar quantities can be combined to form Lumley’s (1970) “flatness” parameter $A = 1 - 9/8(A_2 - A_3)$.

The quantity $A$ is 1 in isotropic turbulence, but zero in 2-component turbulence (where one of the normal stresses vanishes).

These invariants can thus be used to make model sensitive to different types of turbulence structure.

Non-linear elements and anisotropy invariants can also be used to construct “realizable” models. That is, models which cannot return unphysical values such as negative normal stresses.

In a realizable model, the forms employed for $\phi_{ij}$ and $\epsilon_{ij}$ are designed so that the rate of change $Dv^2/Dt \to 0$ when $v^2 \to 0$.

Provided that the first non-zero derivative is also positive, the model will always return values for the normal stress which are greater than or equal to zero.

Models have been developed that include the above features, but these are beyond the scope of the present course.

A circular jet issuing parallel to a wall has applications in film cooling, boundary layer control and in jet engines prior to take-off.

The lateral spreading rate is found to be much greater than that normal to the wall.

This is a result of a strong secondary flow, driven by normal stress anisotropy.

Accurate prediction of the flow thus requires a model capable of returning accurate normal stress values.

Data: Karlsson et al (1992)

TCL results agree well with experiments.

Plane Wall Jet

3-D Wall Jets

Inlet

Wall

Plane Wall Jet

Half-Width Development

Centreline Velocity Decay

Data: Karlsson et al (1992)

3-D Wall Jets

A circular jet issuing parallel to a wall has applications in film cooling, boundary layer control and in jet engines prior to take-off.

The lateral spreading rate is found to be much greater than that normal to the wall.

This is a result of a strong secondary flow, driven by normal stress anisotropy.

Accurate prediction of the flow thus requires a model capable of returning accurate normal stress values.
Predicted fully-developed spreading rates:

<table>
<thead>
<tr>
<th></th>
<th>$dy_{1/2}/dz$</th>
<th>$dx_{1/2}/dz$</th>
<th>$\dot{x}<em>{1/2}/\dot{y}</em>{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expt. (Abrahamsson et al, 1997)</td>
<td>0.065</td>
<td>0.32</td>
<td>4.94</td>
</tr>
<tr>
<td>$k$-$\varepsilon$</td>
<td>0.079</td>
<td>0.069</td>
<td>0.88</td>
</tr>
<tr>
<td>Basic RSM</td>
<td>0.081</td>
<td>0.079</td>
<td>0.97</td>
</tr>
<tr>
<td>Basic RSM + wall ref.</td>
<td>0.053</td>
<td>0.814</td>
<td>15.3</td>
</tr>
<tr>
<td>TCL RSM</td>
<td>0.060</td>
<td>0.51</td>
<td>8.54</td>
</tr>
</tbody>
</table>

- The $k$-$\varepsilon$ model does not capture the asymmetric spreading.
- The RSM's appear to overpredict the spreading rate ratio.
- However, the computed fully-developed state was not attained until very far downstream (400 diameters or more with the TCL model).
- At around $z/D = 70$, the TCL model returns spreading rates of $dy_{1/2}/dz = 0.055$ and $dx_{1/2}/dz = 0.308$, close to experimental values.
Velocity and pressure contours:

The RSM gives the larger accelerated core region.

Streamwise Velocity at $x/c = 0.678$

Fine downstream grid required to resolve vortex development.

Advanced stress transport scheme (TCL) does capture the flow development.

Supersonic Fin-Plate Junction

- Computed by Batten et al (1999)

The multiple vortex structure is not captured by the SST model.

The TCL model (labelled MCL) does produce a reasonably good representation of the flow.

References


