Inviscid Flows
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- Boundary layers
- Transition, Reynolds averaging
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Reading:
- F.M. White, Fluid Mechanics
- J. Mathieu, J. Scott, An Introduction to Turbulent Flow
- P. Libby, Introduction to Turbulence
- S.B. Pope, Turbulent Flows
- D. Wilcox, Turbulence Modelling for CFD

Notes: http://cfd.mace.manchester.ac.uk/tmcfd
- People - T. Craft - Online Teaching Material

Introduction
- One class of flows for which a simplified form of the Navier Stokes equation can be considered is where the viscosity can be neglected.
- These are known as incompressible flows.
- Such flows might arise where the Reynolds number \( Re = UL/ν \) is high.
- Since the Reynolds number represents a ratio of convective to diffusive influences, the limit of very high \( Re \) implies negligible diffusion.
- In high-speed external flows around streamlined bodies, for example, the flow well away from the body might be treated as incompressible (although the approximation is not valid in the boundary layer or wake regions).

The Euler Equations
- If viscous effects are neglected, the governing equations are known as the Euler equations.
- In 2-D, incompressible, flow these can be written as

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} \tag{5}
\]

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial y} \tag{6}
\]
The Stream Function

In 2-D we can define a stream function, \( \psi \), such that the velocity components are given by

\[
U = \frac{\partial \psi}{\partial y} \quad V = -\frac{\partial \psi}{\partial x}
\]

(7)

Note that this definition ensures continuity is satisfied.

From the expressions in equation (7), the stream function \( \psi \) can be evaluated by integrating along a path in the fluid:

\[
\psi = \int_A^B U \, dy - \int_A^B V \, dx
\]

(8)

From equation (7),

\[
U \cdot \nabla \psi = \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \psi = 0
\]

The gradient of \( \psi \) is thus normal to the velocity vector, so \( \psi \) does not change in the direction following the fluid velocity.

Hence \( \psi \) is constant along a streamline.

Contours of \( \psi \) therefore give the streamlines of a flow.

The integral

\[
\int_A^B U \, dy - \int_A^B V \, dx
\]

(9)
gives the total volume flow rate across a line between points \( A \) and \( B \).

The difference in stream function values between two streamlines therefore gives the total volume flow rate between the lines.

Note that the above definition and properties of \( \psi \) are not restricted to inviscid flows.

In steady inviscid flow, subtracting \( \frac{\partial}{\partial x} \) of equation (4) (\( V \) momentum) from \( \frac{\partial}{\partial y} \) of equation (3) (\( U \) momentum) leads to a partial differential equation for \( \psi \):

\[
\frac{\partial}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \psi) + \frac{\partial}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \psi) = 0
\]

(10)

where \( \nabla^2 \) is the Laplacian operator \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

In principle this can be solved (with appropriate boundary conditions), although numerical solution methods are usually required.

Having obtained \( \psi \), the velocity components can then be calculated from equation (7).

Vorticity

In a 2-D flow the vorticity, \( \omega_z \), is given by

\[
\omega_z = \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}
\]

(11)

This is associated with rotation of a fluid element.

In inviscid flow there are no viscous shear stresses to deform fluid elements. It might, therefore, seem a reasonable approximation to assume that the flow will be irrotational, ie. \( \omega_z = 0 \).

This is an approximation that can often be made in “free stream” regions, away from boundary layers.
One can show that in an incompressible, inviscid, flow the vorticity is constant along a streamline.

Taking \( \partial / \partial y \) of the U momentum equation (5) gives
\[
\frac{\partial}{\partial t} \left[ \frac{\partial U}{\partial y} \right] + U \frac{\partial}{\partial x} \left[ \frac{\partial U}{\partial y} \right] + \frac{\partial U}{\partial y} \frac{\partial U}{\partial x} + V \frac{\partial}{\partial y} \left[ \frac{\partial U}{\partial y} \right] + \frac{\partial V}{\partial y} \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} \tag{12}
\]

Since \( \partial U / \partial x + \partial V / \partial y = 0 \) by continuity, we obtain
\[
\frac{\partial}{\partial t} \left[ \frac{\partial U}{\partial y} \right] + U \frac{\partial}{\partial x} \left[ \frac{\partial U}{\partial y} \right] + V \frac{\partial}{\partial y} \left[ \frac{\partial U}{\partial y} \right] = - \frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} \tag{13}
\]

Similarly, taking \( \partial / \partial x \) of the V momentum equation results in
\[
\frac{\partial}{\partial t} \left[ \frac{\partial V}{\partial x} \right] + U \frac{\partial}{\partial x} \left[ \frac{\partial V}{\partial x} \right] + V \frac{\partial}{\partial y} \left[ \frac{\partial V}{\partial x} \right] = - \frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} \tag{14}
\]

Subtracting equation (14) from equation (13) one obtains
\[
\frac{\partial \omega_z}{\partial t} + U \frac{\partial \omega_z}{\partial x} + V \frac{\partial \omega_z}{\partial y} = 0 \quad \text{or} \quad \frac{D \omega_z}{Dt} = 0 \tag{15}
\]

Recall that the total derivative \( D/\text{Dt} \) represents the total rate of change as a result of being convected with the flow.

Hence, if such a flow starts off irrotational (eg. at some far upstream location where the flow is uniform, so \( \omega_z = 0 \)) it must remain irrotational.

From the definition of \( \psi \),
\[
\omega_z = \frac{\partial U}{\partial y} \quad \frac{\partial V}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{16}
\]

The condition \( \omega_z = 0 \) thus implies that the stream function satisfies the Laplace equation, \( \nabla^2 \psi = 0 \).

Note the correct wall boundary condition for an inviscid flow is not simply all velocity components being zero.

The wall-normal velocity component should be zero at a wall, but since there is no viscosity the wall-tangential velocity need not be – its value arises as part of the problem solution.

From the (steady) U momentum equation, noting that \( \partial V / \partial x = \partial U / \partial y \) in irrotational flow, we get
\[
- \frac{1}{\rho} \frac{\partial p}{\partial x} = U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U \frac{\partial U}{\partial x} + V \frac{\partial V}{\partial x} = \frac{\partial}{\partial x} \left[ 0.5(U^2 + V^2) \right] \tag{19}
\]

Similarly, from the V momentum equation one gets
\[
- \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial}{\partial y} \left[ 0.5(U^2 + V^2) \right] \tag{20}
\]

Integrating the above equations we obtain
\[
P + 0.5\rho(U^2 + V^2) = \text{Constant} \tag{21}
\]

For analyzing irrotational, inviscid, flow the velocity potential function, \( \phi \) is often used.

This is defined so that
\[
U = \frac{\partial \phi}{\partial x} \quad V = \frac{\partial \phi}{\partial y} \quad \text{and, in 3-D,} \quad W = \frac{\partial \phi}{\partial z} \tag{17}
\]

Note that this ensures \( \omega_z = \partial U / \partial y - \partial V / \partial x = 0 \).

Continuity then dictates that \( \phi \) satisfies the Laplace equation:
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{18}
\]

With appropriate boundary conditions, the above equation can be solved to obtain the flowfield for a particular problem.
This is a special case of Bernoulli’s equation. The right hand side of equation (21) is now constant throughout the flow, not simply along a streamline.

Having obtained \( \phi \) (and hence the velocities) from solving equation (18), the pressure field can then be determined from equation (21).

An attraction of the above approach is that the Laplace equation is relatively easy to solve. There are some known analytical solutions, and well-tested and inexpensive methods for numerical solutions.

The use of the velocity potential for irrotational flows can easily be extended to 3-D flows, whereas the stream function cannot.

Example 1: Plane Channel Flow

Consider a channel flow developing between two parallel plates.

Appropriate boundary conditions are:

- \( V = 0 \), so \( \partial \phi / \partial y = 0 \), at \( y = 0 \) and 1.
- \( U = U_0 \), so \( \partial \phi / \partial x = U_0 \), at \( x = 0 \).
- \( \partial U / \partial x = 0 \), so \( \partial^2 \phi / \partial x^2 = 0 \), as \( x \to \infty \).

The function \( \phi = U_0 x \) satisfies the Laplace equation, and the above boundary conditions.

Differentiating \( \phi \) gives the velocity field as \( U = U_0 \) and \( V = 0 \).

Potential contour lines and streamlines are as shown.

Without viscosity there is no boundary layer development.

Example 2: Plane Stagnation Flow

In this case it is convenient to work in polar coordinates \( (r, \theta) \).

The Laplace equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0
\]  

(22)

There are a family of solutions to this of the form

\[
\phi = Ar^n \cos(n\theta)
\]

(23)

For simplicity, take \( A = 1 \).

- \( n = 1 \) gives uniform flow.
- \( n = 2 \) gives a flow impinging onto a flat plate, with potential lines and streamlines as shown.
In cylindrical polar coordinates the velocity components are given by

\[ V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad \text{and} \quad V_r = \frac{\partial \phi}{\partial r} \]  
(24)

The radial velocity, \( V_r \), in this flow is thus

\[ \frac{\partial \phi}{\partial r} = 2r \cos(2\theta) \]  
(25)

At \( \theta = 90^\circ \), this gives \( \partial \phi / \partial r = -2r \)
- So the velocity along the stagnation line decreases linearly towards the wall.
- Along the impingement wall (\( \theta = 0^\circ \)) we get \( \partial \phi / \partial r = 2r \)
  - So the velocity along the wall increases linearly with distance from the stagnation point.
- Flow deflections through angles other than \( 90^\circ \) can be obtained by taking other values of \( n \).
Superposition of Solutions

- Since the Laplace equation is linear, solutions to it can be added together, or superimposed.

- For example, the potential function $\phi = k \theta$, for constant $k$, gives a vortex centered at the origin.

- This could be added to the function used in example 3, giving flow around a cylinder with circulation:

$$\phi = U_o(r + R^2/r) \cos(\theta) + k \theta$$

Summary

- In 2-D flows the stream function can be defined to aid in flow analysis.
- Inviscid flows are governed by the Euler equations.
- In an inviscid, irrotational, flow the velocity potential function can be defined.
- In potential flow problems the streamlines and potential function contours are orthogonal to each other.
- Inviscid flow approximations can typically be applied to high speed flow away from solid surfaces. Such solutions can, in some instances, be matched to approximate boundary layer solutions for the near-wall region.