School of Mechanical Aerospace and Civil Engineering

3rd Year Fluid Mechanics

# The Navier Stokes Equations

T. J. Craft

George Begg Building, C41

Contents:



Reading:

F.M. White, *Fluid Mechanics*
J. Mathieu, J. Scott, *An Introduction to Turbulent Flow*
P.A. Libby, *Introduction to Turbulence*
P. Bernard, J. Wallace, *Turbulent Flow: Analysis Measurement & Prediction*
S.B. Pope, *Turbulent Flows*
D. Wilcox, *Turbulence Modelling for CFD*

Notes: http://cfd.mace.manchester.ac.uk/tmcfd
- People - T. Craft - Online Teaching Material

## Derivation of The Navier Stokes Equations

- Here, we outline an approach for obtaining the Navier Stokes equations that builds on the methods used in earlier years of applying mass conservation and force-momentum principles to a control volume.
- The approach involves:
  - Defining a small control volume within the flow.
  - Applying the mass conservation and force-momentum principle to the control volume.
  - Considering what happens in the limit as the control volume becomes infinitesimally small.
- Although the derivation can be done using any arbitrarily shaped control volume, for simplicity we consider here a rectangular control volume.
- We will first derive the equations for two-dimensional, unsteady, flow conditions, and it should then be apparent how these extend to three-dimensional flows.

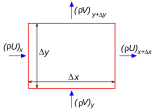


## Mass Conservation (Continuity)

- The mass conservation principle is

$$\begin{bmatrix}\text{Rate of mass accu-}\\ \text{mulation within CV}\end{bmatrix} = \begin{bmatrix}\text{Rate of mass}\\ \text{flow into CV}\end{bmatrix} - \begin{bmatrix}\text{Rate of mass}\\ \text{flow out of CV}\end{bmatrix}$$

- For a two-dimensional control volume of dimensions $\Delta x$ and $\Delta y$ as shown:




Mass accumulation rate $= \partial(\rho \Delta x \Delta y)/\partial t$

Mass inflow $= (\rho U)_x \Delta y + (\rho V)_y \Delta x$

Mass outflow $= (\rho U)_{x+\Delta x} \Delta y + (\rho V)_{y+\Delta y} \Delta x$

- The mass conservation equation thus gives

$$\frac{\partial(\rho \Delta x \Delta y)}{\partial t} = (\rho U)_x \Delta y + (\rho V)_y \Delta x - (\rho U)_{x+\Delta x} \Delta y - (\rho V)_{y+\Delta y} \Delta x \qquad (1)$$

- Division by $\Delta x \Delta y$ and rearrangement leads to

$$\frac{\partial \rho}{\partial t} = \frac{(\rho U)_x - (\rho U)_{x+\Delta x}}{\Delta x} + \frac{(\rho V)_y - (\rho V)_{y+\Delta y}}{\Delta y} \qquad (2)$$



- In the limit as $\Delta x, \Delta y \to 0$, the control volume becomes infinitesimally small, and using Taylor series expansions we have

$$(\rho U)_{x+\Delta x} \to (\rho U)_x + \Delta x \frac{\partial(\rho U)}{\partial x} \qquad (\rho V)_{y+\Delta y} \to (\rho V)_y + \Delta y \frac{\partial(\rho V)}{\partial y}$$

- Substituting these into equation (2) gives

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho U)}{\partial x} - \frac{\partial(\rho V)}{\partial y} \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \qquad (3)$$

- In three-dimensional flows this is easily extended to

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0 \qquad (4)$$

- In steady-state flows, $\partial \rho/\partial t = 0$, so

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0 \qquad (5)$$

- In incompressible flows the density is constant, so we obtain

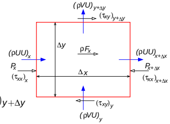
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \qquad (6)$$

## Force-Momentum Principle

- The force-momentum principle is

$$\begin{bmatrix}\text{Accumulation of mo-}\\\text{mentum within CV}\end{bmatrix} = \begin{bmatrix}\text{Rate of momen-}\\\text{tum flow into CV}\end{bmatrix} - \begin{bmatrix}\text{Rate of momen-}\\\text{tum flow out of CV}\end{bmatrix} + \begin{bmatrix}\text{Forces acting}\\\text{on CV faces}\end{bmatrix} + \begin{bmatrix}\text{Body forces}\\\text{within CV}\end{bmatrix}$$

- Consider the $U$ momentum equation, on a control volume of dimensions $\Delta x$ and $\Delta y$:




Accumulation rate $= \Delta x \Delta y \left[\partial(\rho U)/\partial t\right]$

Mom. flux in $= \Delta y(\rho UU)_x + \Delta x(\rho VU)_y$

Mom. flux out $= \Delta y(\rho UU)_{x+\Delta x} + \Delta x(\rho VU)_{y+\Delta y}$

- Surface forces arise from the pressure and viscous stresses:

Net surface force $= \left[(P+\tau_{xx})_x - (P+\tau_{xx})_{x+\Delta x}\right]\Delta y + \left[(\tau_{xy})_y - (\tau_{xy})_{y+\Delta y}\right]\Delta x$

Body force $= \rho F_x \Delta x \Delta y$



- The $U$-momentum balance then gives

$$\Delta x \Delta y\left[\partial(\rho U)/\partial t\right] = \left[(\rho UU)_x - (\rho UU)_{x+\Delta x}\right]\Delta y + \left[(\rho VU)_y - (\rho VU)_{y+\Delta y}\right]\Delta x + \left[(P+\tau_{xx})_x - (P+\tau_{xx})_{x+\Delta x}\right]\Delta y + \left[(\tau_{xy})_y - (\tau_{xy})_{y+\Delta y}\right]\Delta x + \rho F_x \Delta x \Delta y \quad (7)$$

- Dividing by $\Delta x \Delta y$ gives:

$$\frac{\partial(\rho U)}{\partial t} = \frac{(\rho UU)_x - (\rho UU)_{x+\Delta x}}{\Delta x} + \frac{(\rho VU)_y - (\rho VU)_{y+\Delta y}}{\Delta y} + \frac{P_x - P_{x+\Delta x}}{\Delta x} + \frac{(\tau_{xx})_x - (\tau_{xx})_{x+\Delta x}}{\Delta x} + \frac{(\tau_{xy})_y - (\tau_{xy})_{y+\Delta y}}{\Delta y} + \rho F_x \quad (8)$$

- As before, as $\Delta x$ and $\Delta y \to 0$, for any quantity $\phi$ we have:

$$\phi_{x+\Delta x} \to \phi_x + \Delta x \frac{\partial \phi}{\partial x} \quad \text{and} \quad \phi_{y+\Delta y} \to \phi_y + \Delta y \frac{\partial \phi}{\partial y}$$

- Applying these to equation (8) the $U$-momentum balance becomes

$$\frac{\partial(\rho U)}{\partial t} = -\frac{\partial(\rho UU)}{\partial x} - \frac{\partial(\rho VU)}{\partial y} - \frac{\partial P}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} + \rho F_x \quad (9)$$



- Rearranging gives the usual form of the $U$-momentum equation:

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{\partial(\rho VU)}{\partial y} = -\frac{\partial P}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} + \rho F_x \quad (10)$$

- As with the continuity equation, the $U$ momentum equation is also a differential equation.

- The corresponding $V$-momentum equation is

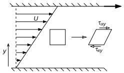
$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{\partial(\rho V^2)}{\partial y} = -\frac{\partial P}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} + \rho F_y \quad (11)$$

- In their above forms, however, the $U$ and $V$-momentum equations still contain additional unknown variables, namely the viscous stresses, $\tau_{xx}$, $\tau_{yy}$ and $\tau_{xy}$.



## The Viscous Stresses

- In a simple shear flow, Stoke's law states that the viscous shear stress, $\tau_{xy}$, is obtained from



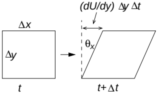

$$\tau_{xy} = -\mu \frac{\partial U}{\partial y}$$

- This equation can be obtained by considering how, in a simple case, the rate at which a fluid element is deformed is opposed by the fluid viscosity.

$$\tan(\theta_x) = \frac{(\partial U/\partial y)\Delta y \Delta t}{\Delta y}$$

For small $\theta_x$, $\tan(\theta_x) \approx \theta_x$, so

$$\frac{\partial \theta_x}{\partial t} \approx \frac{\partial U}{\partial y}$$




- For many common fluids we have $\tau \propto \partial \theta_x / \partial t$.

- In the more general case, expressions for the viscous stresses can again be derived by considering the deformation caused by the flow field to an initially rectangular fluid element.
- For Newtonian fluids these general stress-strain relations can be expressed as the viscous stresses being linearly related to the strain rates, with the constant of proportionality being the viscosity $\mu$.
- Hence, in 2-D, we obtain the viscous stresses as

$$\tau_{xx} = -2\mu\frac{\partial U}{\partial x} \qquad \tau_{yy} = -2\mu\frac{\partial V}{\partial y} \qquad \tau_{xy} = -\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

- Substituting the expressions for $\tau_{xx}$ and $\tau_{xy}$ into the $U$ momentum equation gives:

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{\partial(\rho VU)}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] + \rho F_x \quad (12)$$

- A similar equation can be derived for the $V$ momentum component.



## The Navier Stokes Equations

- The above set of equations that describe a real fluid motion are collectively known as the **Navier Stokes equations**. In 2-D they can be written as:

**The continuity equation:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \quad (13)$$

**The $U$-momentum equation:**

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho U^2)}{\partial x} + \frac{\partial(\rho VU)}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] + \rho F_x \quad (14)$$

**The $V$-momentum equation:**

$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{\partial(\rho V^2)}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial V}{\partial y}\right] + \rho F_y \quad (15)$$



- In three-dimensional flows the equations are expanded to:

**Continuity:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} + \frac{\partial(\rho W)}{\partial z} = 0 \quad (16)$$

**$U$-Momentum:**

$$\frac{\partial(\rho U)}{\partial t} + \frac{\partial(\rho UU)}{\partial x} + \frac{\partial(\rho VU)}{\partial y} + \frac{\partial(\rho WU)}{\partial z} = -\frac{\partial P}{\partial x} + \rho F_x + 2\frac{\partial}{\partial x}\left[\mu\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\right] \quad (17)$$

**$V$-Momentum:**

$$\frac{\partial(\rho V)}{\partial t} + \frac{\partial(\rho UV)}{\partial x} + \frac{\partial(\rho VV)}{\partial y} + \frac{\partial(\rho WV)}{\partial z} = -\frac{\partial P}{\partial y} + \rho F_y + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right] + 2\frac{\partial}{\partial y}\left[\mu\frac{\partial V}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\right] \quad (18)$$

**$W$-Momentum:**

$$\frac{\partial(\rho W)}{\partial t} + \frac{\partial(\rho UW)}{\partial x} + \frac{\partial(\rho VW)}{\partial y} + \frac{\partial(\rho WW)}{\partial z} = -\frac{\partial P}{\partial z} + \rho F_z + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)\right] + 2\frac{\partial}{\partial z}\left[\mu\frac{\partial W}{\partial z}\right] \quad (19)$$



## Convection and Diffusion Terms

- The term

$$\frac{\partial(\rho U\phi)}{\partial x} + \frac{\partial(\rho V\phi)}{\partial y} + \frac{\partial(\rho W\phi)}{\partial z}$$

where $\phi$ stands for any of the velocity components ($U$, $V$ or $W$) represents convection of $\phi$ by the fluid.

- The terms on the right hand sides of the equations involving the viscosity represent viscous diffusion.
- The general form of the momentum transport equations is thus seen to be

$$\text{Time derivative} + \text{Convection terms} = \text{Forcing terms} + \text{Diffusion terms}$$

- The combination of time derivative and convection terms represents the total rate of change of a quantity following a fluid path line. It is often written in shorthand notation as $D\phi/Dt$:

$$\frac{D\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + \frac{\partial(U\phi)}{\partial x} + \frac{\partial(V\phi)}{\partial y} + \frac{\partial(W\phi)}{\partial z} \quad (20)$$

- The time derivative and convection terms are sometimes written as above (with $\rho$, $U$, $V$, $W$ inside the derivatives), and sometimes in the form

$$\rho\frac{\partial\phi}{\partial t}+\rho U\frac{\partial\phi}{\partial x}+\rho V\frac{\partial\phi}{\partial y}+\rho W\frac{\partial\phi}{\partial z}$$

- These are, in fact, entirely equivalent, since differentiating by parts gives

$$\begin{aligned}&\frac{\partial\rho\phi}{\partial t}+\frac{\partial\rho U\phi}{\partial x}+\frac{\partial\rho V\phi}{\partial y}+\frac{\partial\rho W\phi}{\partial z}\\&=\rho\frac{\partial\phi}{\partial t}+\phi\frac{\partial\rho}{\partial t}+\rho U\frac{\partial\phi}{\partial x}+\phi\frac{\partial\rho U}{\partial x}+\rho V\frac{\partial\phi}{\partial y}+\phi\frac{\partial\rho V}{\partial y}+\rho W\frac{\partial\phi}{\partial z}+\phi\frac{\partial\rho W}{\partial z}\\&=\rho\frac{\partial\phi}{\partial t}+\rho U\frac{\partial\phi}{\partial x}+\rho V\frac{\partial\phi}{\partial y}+\rho W\frac{\partial\phi}{\partial z}+\phi\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho U}{\partial x}+\frac{\partial\rho V}{\partial y}+\frac{\partial\rho W}{\partial z}\right)\end{aligned}$$

and the term in brackets multiplied by $\phi$ is zero from the continuity equation.



- If the viscosity is constant the diffusion terms can be simplified by taking $\mu$ outside the derivatives. In 2-D, for example:

$$\begin{aligned}&2\frac{\partial}{\partial x}\left[\mu\frac{\partial U}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right]\\&=\mu\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right]+\mu\frac{\partial}{\partial y}\left[\frac{\partial U}{\partial y}\right]+\mu\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right]+\mu\frac{\partial}{\partial y}\left[\frac{\partial V}{\partial x}\right]\\&=\mu\frac{\partial^2U}{\partial x^2}+\mu\frac{\partial^2U}{\partial y^2}+\mu\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right]\\&=\mu\frac{\partial^2U}{\partial x^2}+\mu\frac{\partial^2U}{\partial y^2}\end{aligned}$$



## Further Simplifications

- In many flows that we will consider certain additional simplifications can be introduced.
- In steady flows the time derivatives become zero:

$$\partial(\rho U)/\partial t=\partial(\rho V)/\partial t=\partial(\rho W)/\partial t=0$$

- The body force terms, $F_x$, $F_y$, $F_z$, are, in many cases, negligible.
- These simplifications lead to the momentum equations for a 2-D steady, incompressible, constant viscosity, flow without body forces being given by

$$\rho\frac{\partial(U^2)}{\partial x}+\rho\frac{\partial(VU)}{\partial y}=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2U}{\partial x^2}+\mu\frac{\partial^2U}{\partial y^2}$$

$$\rho\frac{\partial(UV)}{\partial x}+\rho\frac{\partial(V^2)}{\partial y}=-\frac{\partial P}{\partial y}+\mu\frac{\partial^2V}{\partial x^2}+\mu\frac{\partial^2V}{\partial y^2}$$



## Other Transport Equations

- The governing equations for other quantities transported by a flow often take the same general form of transport equation to the above momentum equations.
- For example, the transport equation for the evolution of temperature in a fluid flow can often be written (in 2-D for simplicity) as

$$\frac{\partial T}{\partial t}+\frac{\partial(UT)}{\partial x}+\frac{\partial(VT)}{\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha\frac{\partial T}{\partial y}\right)$$

where $\alpha$ is the molecular thermal diffusivity.

- Notice the general form of

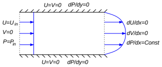
Time derivative $+$ Convection terms $=$ Diffusion terms $+$ Source terms

(in this case, the source, or forcing, terms are zero).

- We will meet transport equations for other, turbulence-related, quantities later in the course.

## Solving the Navier Stokes Equations

- The Navier Stokes equations form a system of differential equations:
  - In two-dimensional flows there are three variables ($U$, $V$, $P$) and three differential equations (Continuity, $U$ and $V$-momentum).
  - In three-dimensional flows there are four variables and four differential equations.
- Although the equations have been presented for a Cartesian coordinate system ($x$, $y$, $z$), they can also be transformed mathematically to other coordinate systems, (eg. cylindrical, or spherical, polars).
- In principle, therefore, the Navier Stokes equations can be integrated over a flow domain of interest, with appropriate boundary conditions, to produce detailed velocity and pressure fields.




- Although analytical solutions can be obtained for a few cases, in practice the equations must usually be solved using numerical methods.



## Tensor/Summation Notation

- Although it is sometimes appropriate to write the Navier Stokes equations out in their expanded Cartesian form as above, in general it becomes rather cumbersome.
- Tensor notation and, in particular, the Einstein summation convention is often used to write the equations in a more compact, shorthand, form.
- One can use subscripts to denote the elements of a vector or tensor.
- The summation convention means that if a subscript letter is repeated in an expression, there is an implied summation over it.
- So, for example,

$$\frac{\partial U_i}{\partial x_i} \equiv \sum_{i=1,3} \frac{\partial U_i}{\partial x_i} = \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3}$$

- The continuity equation for incompressible flow can then simply be written as

$$\partial U_i / \partial x_i = 0 \tag{21}$$

- Note that the "$i$" in the above expression could have been replaced by "$j$" or "$k$" (or anything else). It is purely a dummy index indicating summation.



- Using the same notation, the three momentum equations from the Navier Stokes system can be written compactly as

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial \rho U_i U_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j}\right) + \rho F_i \tag{22}$$

- Here, the 2nd and 4th terms contain a repeated "$j$", so one sums from $j$ equals 1 to 3 in them. For example, the convection term expands to

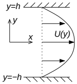
$$\frac{\partial \rho U_i U_j}{\partial x_j} \equiv \frac{\partial \rho U_i U_1}{\partial x_1} + \frac{\partial \rho U_i U_2}{\partial x_2} + \frac{\partial \rho U_i U_3}{\partial x_3}$$

- The subscript $i$ is not repeated, and is being used to denote a component of the velocity vector (one gets the $U_1$ momentum equations by setting $i = 1$, the $U_2$ one by setting $i = 2$, etc).
- Equations (21) and (22) are far more compact and convenient than using the expansions of equations (16) to (19) for the Navier Stokes system.
- For much of this course we can relatively easily write equations out in terms of $x$, $y$ components etc., and will not have to use summation notation. It is widely used in textbooks and papers on fluid mechanics and turbulence, and we will use it in a few places, for convenience.



## Analytical Solutions of The Navier Stokes Equations

- There are a few, very simple, laminar flows for which the Navier Stokes equations can be solved analytically.
- For example, for steady, incompressible, fully developed flow in a plane channel as shown, we have $V = 0$ and $U$ does not depend on $x$.




- Continuity ($\partial U/\partial x + \partial V/\partial y = 0$) is satisfied.
- The $V$ momentum equation reduces to $\partial P/\partial y = 0$, so $P$ is constant across the channel.
- The $U$ momentum equation becomes

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) \tag{23}$$

with boundary conditions $U = 0$ at $y = \pm h$.

- Since $P$ is not a function of $y$, we can easily integrate this:

$$\mu \frac{\partial U}{\partial y} = y \frac{\partial P}{\partial x} + A \tag{24}$$

for some constant of integration $A$. Integrating a second time gives

$$\mu U = \frac{y^2}{2} \frac{\partial P}{\partial x} + Ay + B \tag{25}$$

- To determine the constants $A$ and $B$, we apply the boundary conditions that $U = 0$ at $y = \pm h$:

$$0 = \frac{h^2}{2} \frac{\partial P}{\partial x} + Ah + B \qquad \text{and} \qquad 0 = \frac{h^2}{2} \frac{\partial P}{\partial x} - Ah + B \tag{26}$$

This gives

$$A = 0 \qquad \text{and} \qquad B = -\frac{h^2}{2} \frac{\partial P}{\partial x} \tag{27}$$

- The velocity profile is therefore given by the parabola

$$U = -\frac{1}{2\mu} \frac{\partial P}{\partial x} (h^2 - y^2) \tag{28}$$

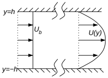


- The pressure gradient can be related to the bulk (average) velocity, since

$$U_b = \frac{1}{2h} \int_{-h}^{h} U(y) dy = -\frac{1}{4h\mu} \frac{\partial P}{\partial x} \int_{-h}^{h} (h^2 - y^2) dy$$
$$= -\frac{1}{4h\mu} \frac{\partial P}{\partial x} \left[ h^2 y - y^3/3 \right]_{-h}^{h} = -\frac{h^2}{3\mu} \frac{\partial P}{\partial x}$$

- Hence $\partial P / \partial x = -3\mu U_b / h^2$, and the velocity profile can be written as

$$U = (3/2) U_b (1 - y^2/h^2) \tag{29}$$




- A similar analysis can be applied to some other simple 1-D flows, such as fully-developed pipe flow, flow between moving infinite flat plates, etc.