Introduction

- In a turbulent flow the mean temperature equation can be written as

\[
\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x_j} (U_j \Theta) = \frac{\partial}{\partial x_j} \left( \alpha \frac{\partial \Theta}{\partial x_j} - \overline{u_i \theta} \right)
\]  

(1)

where \( \alpha = \nu / \sigma \) is the thermal diffusivity, with \( \sigma \) the molecular Prandtl number.

- To close the system, we need to approximate the turbulent heat fluxes, \( \overline{u_i \theta} \).

- Here we consider relatively simple methods for approximating \( \overline{u_i \theta} \), which mirror the eddy-viscosity approach for modelling the Reynolds stresses, \( \overline{u_i u_j} \).

- A similar situation arises for other transported scalars in turbulent flows (e.g., species concentration). Similar modelling practices to those outlined here are usually adopted for them.

Eddy-Diffusivity Models

- When considering the dynamic field, an analogy was drawn between the Reynolds stresses and viscous stresses.

- This led to the eddy-viscosity approach of modelling

\[
\overline{u_i u_j} = -\nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + (2/3)k \delta_{ij}
\]

(2)

with \( \nu_t \) being the eddy (or turbulent) viscosity.

- We noted that \( \nu_t \) is not a property of the fluid, but depends on local flow conditions.

- Apart from in the viscous sublayer, \( \nu_t \) is generally much larger than \( \nu \).

- We examined a number of modelling approaches for approximating \( \nu_t \), ranging from mixing-length (zero-equation) schemes to two-equation models.

- Extending the above ideas to the turbulent heat fluxes, we might approximate these by

\[
\overline{u_i \theta} = -\alpha_t \frac{\partial \Theta}{\partial x_i}
\]

(3)

where the eddy-diffusivity, \( \alpha_t \), is taken as \( \alpha_t = \nu_t / \sigma_t \) and \( \sigma_t \) is the turbulent Prandtl number.

- Again, \( \sigma_t \) is not a property of the fluid, but in principle depends on the local flow and turbulence conditions.

- In practice, \( \sigma_t \) is usually taken as a constant of around 0.9 for near-wall flows. In free flows a slightly lower value (around 0.7) is often more appropriate.

- Notice that with this form the turbulent heat transport depends directly on the corresponding mean temperature gradient. A constant temperature in one direction implies no turbulent heat transport in that direction.
In a simple shear flow, with $U(y)$, $\Theta(y)$, the eddy-diffusivity model gives

$$\overline{u\theta} = -\left(\nu_t/\sigma_t\right) \frac{\partial \Theta}{\partial x} = 0$$

$$\overline{v\theta} = -\left(\nu_t/\sigma_t\right) \frac{\partial \Theta}{\partial y}$$

In equilibrium conditions experiments suggest $\sigma_t \approx 0.7$ – 0.8 and $|\overline{u\theta}/\overline{v\theta}| \approx 1.1$ in an homogeneous free shear flow.

Measurements at higher strain rates show $\sigma_t \approx 1.1$ and $|\overline{u\theta}/\overline{v\theta}| \approx 2.2$.

Clearly the model prediction of $\overline{u\theta} = 0$ is not an accurate representation of reality.

However, if streamwise gradients are relatively small, misrepresenting $\overline{u\theta}$ may not have too serious an effect, at least in some non-buoyant flows.

The reason for this can be seen from the boundary layer form of the mean temperature equation:

$$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \frac{\partial}{\partial y} \left( \alpha \frac{\partial \Theta}{\partial y} \overline{v\theta} \right)$$

In this situation only the cross-stream heat flux, $\overline{v\theta}$, is particularly influential.

In a buoyancy-affected flow we saw an extra generation term appears in the $k$ transport equation:

$$G_k = -\beta g_i u_i \overline{\theta}$$

With the above eddy-diffusivity model we relate the turbulent heat fluxes directly to the corresponding temperature gradients.

To understand the model behaviour we thus examine two cases: one where the temperature gradient and gravitational vector are aligned, and one where they are not.

In the (stable) situation shown, using the eddy-diffusivity model for $\overline{v\theta}$, we now get

$$G_k = -\beta g \overline{v\theta} = \beta g (\nu_t/\sigma_t) \frac{\partial \Theta}{\partial y}$$

$\partial \Theta/\partial y$ is positive, $g$ is negative, so $G_k$ is negative as we would expect.

If the temperature gradient were reversed, $G_k$ would change sign, again as expected.

Although qualitatively showing the correct sign, such simple models are often not particularly accurate in a quantitative sense, particularly in stably stratified flows.
Vertical Buoyant Flows

- Now consider a vertical flow as shown.
- We still have
  \[ G_k = -\beta g \nabla \bar{\theta} = \beta g (\nu_t / \sigma_t) \frac{\partial \Theta}{\partial y} \]  \( \text{(7)} \)
- However, \( \frac{\partial \Theta}{\partial y} \) is now rather small (the dominant temperature gradient is normal to the wall), so \( G_k \) is also small.
- From the simple shear flow discussion earlier, we expect the model to underpredict the magnitude of the streamwise heat flux (\( \bar{v} \bar{T} \) in this case).
- Here, this underprediction can be expected to lead to an underestimation of the magnitude of \( G_k \).

The GGDH Heat Flux Model

- An improved turbulent heat flux model is often provided by the generalized gradient diffusion model of Daly & Harlow (1970).
- In this, we take
  \[ \bar{u}_i \bar{\theta} = -c_\theta k \frac{\partial \Theta}{\partial y} \bar{u}_j \frac{\partial \Theta}{\partial x_j} \]  \( \text{(8)} \)
- The constant \( c_\theta \) is typically taken around 0.3.
- In a simple shear flow considered earlier, we now have
  \[ \bar{u} \bar{\theta} = -c_\theta k \frac{\partial \Theta}{\partial y} \]  \[ \bar{v} \bar{\theta} = -(2/3)c_\theta k \frac{\partial \Theta}{\partial y} \]
- Note that reliable values of the individual Reynolds stress components are now needed.

If using the linear EVM formulation for the stresses the heat flux expressions become

- The GGDH generally gives a better representation of the turbulent heat fluxes than the simple eddy-diffusivity model (in the above example, \( \bar{u} \bar{\theta} \) is non-zero now).
- However, to realize a significant improvement a better underlying model for the Reynolds stress components than the eddy-viscosity representation is often needed.