Near-Wall Modelling
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- Navier-Stokes equations
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Reading:
- F.M. White, Fluid Mechanics
- J. Mathieu, J. Scott, An Introduction to Turbulent Flow
- P.A. Libby, Introduction to Turbulence
- S.B. Pope, Turbulent Flows
- D. Wilcox, Turbulence Modelling for CFD
Notes: http://cfd.mace.manchester.ac.uk/tmcfd
- People - T. Craft - Online Teaching Material

Introduction

- The near-wall, viscosity-affected, layer of a turbulent fluid flow poses a number of challenges, from both numerical and modelling viewpoints.
- Viscous effects on the turbulence need to be accounted for – often by including ‘near-wall damping’ terms and other source terms in the modelled transport equations.
- Since very steep gradients of velocity (and turbulence statistics) occur across the viscous layer, very fine grids are needed to provide adequate numerical resolution.
- In this lecture we briefly consider two alternative strategies for modelling near-wall flow regions:
  - Low-Reynolds-number modelling, where the viscous layer is resolved numerically, and viscous effects included in the turbulence model.
  - Wall functions, where the viscous layer is not resolved, but approximations are introduced to account for the flow behaviour across it.

Low-Reynolds-Number Modelling

- In mixing-length and one-equation models, a lengthscale increasing linearly with wall-distance ensured a logarithmic velocity profile in the fully-turbulent near-wall region.
- Damping terms were introduced to reduce the lengthscale (and hence the turbulent viscosity) across the viscous sublayer.
- In the mixing-length model the Van-Driest damping function was introduced:
  \[ l_m = D \min (xy, \lambda \delta) \]  
  \[ D = 1 - \exp (-y^+ / A^+) \]  
  with \( A^+ = 26 \).

Low-Reynolds-Number Two-Equation Models

- When studying two-equation models we considered how to tune the various model coefficients to a number of high-Reynolds-number flows (ie. situations found in fully turbulent flow regions).
- Adaptations are typically required to obtain accuracy across the viscosity-affected near-wall layer.
- These often include near-wall damping terms in the turbulent viscosity and other model coefficients (dependent on wall-distance or turbulent Reynolds number, \( R_t = k^2 / (\varepsilon \nu) \)).
- Additional near-wall source terms are sometimes also included in the modelled lengthscale governing equation.
- Some \( \varepsilon \)-based models employ a modified \( \varepsilon \) equation or variable, in order to simplify boundary conditions, and improve near-wall stability.
As an example, the low-Reynolds-number Launder-Sharma (1974) \( k-\varepsilon \) model uses

\[
\frac{D\bar{\varepsilon}}{Dt} = c_{\varepsilon 1}\frac{\bar{\varepsilon} P_{k}}{k} - c_{\varepsilon 2}\frac{\bar{\varepsilon}^{2}}{k} + \frac{\partial}{\partial x_{j}}\left[ (v + v_{t}/\sigma_{\varepsilon}) \frac{\partial \bar{\varepsilon}}{\partial x_{j}} \right] + 2\nu v_{t} \left( \frac{\partial^{2} U_{i}}{\partial x_{j}\partial x_{k}} \right)^{2} \tag{2}
\]

where

\[
\bar{\varepsilon} = \bar{\varepsilon} - 2\nu \left( \frac{\partial k^{1/2}}{\partial x_{j}} \right)^{2} \tag{3}
\]

The turbulent viscosity is taken as

\[
\nu_{t} = c_{\mu} f_{\mu} \frac{k^{2}}{\bar{\varepsilon}} \tag{4}
\]

where \( c_{\mu} = 0.09 \) and

\[
f_{\mu} = \exp \left\{ \frac{-3.4}{(1 + R_{t}/50)^{2}} \right\} \tag{5}
\]

with \( R_{t} = k^{2}/(\bar{\varepsilon}\nu) \).

Wall Functions

- The above modelling refinements allow us to apply the models across the thin viscous sublayer.
- However, the near-wall grid needs to be fine enough to provide an accurate resolution of the flow variables across this layer.
- Typically, this requires ensuring the near-wall node lies at a non-dimensional distance of around \( y^{+} < 1 \).
- In large 3-D calculations, employing such fine grids to resolve the very near-wall regions of the flow can be extremely expensive.
- We thus consider an alternative method of treating the thin near-wall region where viscous effects are influential.
- The treatment is designed to reduce the computational cost associated with resolving this thin layer, and is based on the ideas of local equilibrium simple shear flows already discussed.

The Need for Wall Functions

- In the immediate wall vicinity, even averaged turbulent flow properties undergo extremely abrupt changes.
- In finite volume, or other numerical schemes, the variation of quantities between nodes is approximated using linear, or sometimes higher order, polynomial interpolation.
- Since there are steep wall-normal gradients across the viscosity-affected layer, one must normally use a much finer grid in the direction normal to the wall than in other regions of the flow.
- As an example, 7–10 nodes might be needed to cover a laminar boundary layer; a turbulent one might require 10–12 nodes for the fully turbulent region, plus an additional 7–20 nodes to cover the viscous sublayer (depending on flow and closure model).
The resulting very elongated computational cells can adversely affect convergence rates, so numerical stability may require one to use a reasonably fine grid in other directions also.

Example near-wall grids around a compressor blade.

Three-dimensional grid around a wing-tip geometry:

It is quite possible to spend 70\% or more of the execution time for a complex 3-D computation in simply resolving the near-wall sublayer.

An approach removing the need for such a finely meshed region would thus result in considerable savings in computer resources.

Development of Wall Functions

In a wall function one aims to obtain approximate analytical expressions for the mean velocity distribution across the near-wall layer, thus avoiding the need for a detailed numerical resolution of the sublayer.

In a finite-volume method, integrating the streamwise momentum equation over a cell leads to a balance between net inflow of momentum, pressure forces, and the shear stresses on the faces of the cell.

Shear stresses at the cell faces are then approximated assuming a linear variation of velocity between nodes.

In the near-wall cell shown the shear stress at the north face of the near-wall cell is evaluated as \( \tau_n = \mu (U_3 - U_2)/(y_3 - y_2) \), and the wall shear stress as \( \tau_w = \mu (U_2 - U_1)/(y_2 - y_1) \).

This gives a reasonable approximation of \( \tau_w \) if the node spacing is sufficiently small.

To see how a wall function could be developed to give a reasonable estimate for \( \tau_w \) even if the near-wall cell is relatively large, we first consider a laminar near-wall flow.

If we assume velocity variations in the \( y \) direction are much larger than in other directions, and convective transport is negligible, integrating the streamwise momentum equation twice results in

\[
U = \left( \frac{\tau_w}{\mu} \right) y + \frac{y^2}{2} \frac{\partial P}{\partial x} \tag{6}
\]

For reasonably small \( y \), or if the pressure gradient is small, the approximation noted earlier,

\[
\tau_w = \mu U_2/y_2 \tag{7}
\]

(taking \( U_1 \) and \( y_1 \) as zero) is thus reasonably accurate.

The wall shear stress force \( (\tau_w \delta x \delta z) \) opposing the fluid’s motion can thus be expressed in terms of velocity differences between adjacent nodes.

For larger cell sizes, we might use equation (6) to give

\[
\tau_w = \left[ \mu U_2 - \left( \frac{y_2^3}{2} \right) \frac{\partial P}{\partial x} \right]/y_2 \tag{8}
\]
In a turbulent flow, the velocity profile will only approximate equation (6) across a small part of the laminar sublayer. It will certainly not hold in the fully turbulent region.

For turbulent flow, we thus consider how to devise an expression allowing an estimate of the wall shear stress in terms of the velocity at the near-wall node, even if the grid is not fine enough to properly resolve the velocity profile between the wall and near-wall nodes.

In a local equilibrium turbulent flow, with \( \ell_m = \kappa y \), we found that the velocity in the fully turbulent region satisfies the log-law:

\[
U^+ = \frac{1}{\kappa} \log(Ey^+) \quad (9)
\]

where \( \kappa \) and \( E \) are usually taken as 0.41 and 9 respectively, and

\[
U^+ = U/\left(\frac{\tau_w}{\rho}\right)^{1/2} \quad y^+ = y\left(\frac{\tau_w}{\rho}\right)^{1/2}/\nu
\]

Rearranging this expression, we can obtain:

\[
\frac{\kappa U}{\log(Ey(\tau_w/\rho)^{1/2}/\nu)} = \left(\frac{\tau_w}{\rho}\right)^{1/2}
\]

or, if we assume the conditions hold at the near-wall node,

\[
\tau_w = \rho \left\{ \frac{\kappa U_2}{\log(Ey_2(\tau_w/\rho)^{1/2}/\nu)} \right\}^2
\]

Consequently, in a turbulent flow one could simply use equation (11) in place of equation (7).

Since the log-law is valid in the fully turbulent flow region, the near-wall node (node 2) should now be located not in the viscous layer, where gradients are very steep and rapidly changing, but in the fully turbulent region (where gradients will be significantly smaller).

In terms of non-dimensional wall-distance, this typically amounts to having \( y^+ \) greater than 30 or so at the near-wall node.

With the above approximation, one thus avoids the need to resolve the sublayer with a very fine grid.

For a numerical solution one also needs the effective viscosity at the near-wall node, in order to interpolate its value to the north cell face.

At node 2, we can obtain the effective viscosity by taking the derivative of equation (9)

\[
\frac{\partial U}{\partial y} = \frac{(\tau_w/\rho)^{1/2}}{\kappa y}
\]

Since \( \tau = \tau_w = \mu_t \partial U/\partial y \) in the fully turbulent region, then

\[
\mu_t = \frac{\tau_w}{\partial U/\partial y} = \rho \kappa y (\tau_w/\rho)^{1/2}
\]

Thus, we can place the near-wall node beyond the viscous sublayer and use equation (11) – in an iterative mode, since it is an implicit equation for \( \tau_w \) – to obtain the wall shear stress, and then equation (12) to obtain the turbulent viscosity.

This approach avoids the need for a very fine near-wall grid, and means that in principle one need only use the high-Reynolds-number form of whatever turbulence model is being employed (since the model is not actually applied in the viscosity-affected region).

The above form of wall function can be used with any of the turbulence models examined so far.

However, it does have a number of weaknesses.

One is that equation (12) implies the turbulent viscosity vanishes when the wall shear stress is zero.

It will thus give essentially zero turbulent viscosity near flow reattachment or impingement points despite the fact that, in reality, turbulence levels will be quite high there.

If using at least a one-equation model, where we have values for the turbulent kinetic energy \( k \), a somewhat improved wall function formulation can be adopted.

This improved form is based on broadly the same strategy, but with a slight reformulation of the log-law.
In a local equilibrium boundary layer we had
\[ |\vec{\nabla} \vec{v}| = \frac{\tau_w}{\rho} \quad \text{and} \quad |\vec{\nabla} \vec{v}| / k = c_\mu^{1/2} \]

It follows that
\[ k = \left( \frac{\tau_w}{\rho} \right) / c_\mu^{1/2}, \]
and we could use \( k^{1/2} \) instead of \( \left( \frac{\tau_w}{\rho} \right)^{1/2} \) as a velocity scale in non-dimensionalizing the mean velocity and wall-distance.

The modified non-dimensional velocity and wall-distance are usually written as
\[ U^* = \frac{U k^{1/2}}{\left( \frac{\tau_w}{\rho} \right)} \quad \text{and} \quad y^* = y k^{1/2} / \nu \quad (13) \]
and the corresponding log-law becomes
\[ U^* = \frac{1}{\kappa^*} \log(E^* y^*) \quad (14) \]
with \( \kappa^* = c_\mu^{1/4} \kappa \) and \( E^* = c_\mu^{1/4} E \).

The above form is, in fact, more widely used in general-purpose software than the mixing length form of log-law considered earlier.

With the above alternative formulation, the effective viscosity \( \mu_t \) at the near-wall node is now given by
\[ \mu_t = \rho \kappa^* y k^{1/2}, \]
which does not vanish when \( \tau_w \) does.

In practice, the value of \( k \) appearing in the above expressions is usually simply taken as the value at the near wall node, \( k_2 \).

There are further improvements that can be made:
- Some schemes try to use a value of \( k \) that will be less dependent on the position of the near-wall node.
- Others introduce additional terms to account for situations where the near-wall flow may not be expected to obey the log-law.
- Details of these more advanced schemes are not, however, covered here.