Modelling of the Turbulent Heat Fluxes in Natural, Forced and Mixed Convection Regimes

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Abstract

A new model for the turbulent heat fluxes, the Elliptic-Blending Algebraic Flux Model (EB-AFM), as well as a simplified version, the EB-GGDH, are proposed, in order to incorporate the nonlocal blocking effect of the wall. They are successfully tested in four different 1D cases, ranging from natural to forced convection.

1 Introduction

In industrial applications, most of the problems are treated with the Simple Gradient Diffusion Hypothesis (SGDH) for the heat fluxes, in particular when eddy-viscosity models are used for the Reynolds stress. With the progress of computing resources, Reynolds stress models become affordable, making possible the use of different types of turbulent heat flux closures: Simple Gradient Diffusion Hypothesis (SGDH), Generalized Gradient Diffusion Hypothesis (GGDH), linear Algebraic Flux Models (AFM), Non-Linear Algebraic Flux Model (NL-AFM), Kenjereš \textit{et al.}, 2005 and heat flux transport models.

The GGDH was successfully applied in many flows in the forced-convection regime (e.g., Behnia \textit{et al.}, 1998; Manceau \textit{et al.}, 2000; Sveningsson & Davidson, 2005; Thielen \textit{et al.}, 2005), provided that the model for the Reynolds stress correctly reproduces the anisotropy and the near-wall balance (Lauder, 1975; Hanjalić, 2002). Since complex, industrial flows can exhibit all the convection regimes, a reasonable compromise between simplicity, numerical robustness and representation of the physics must be sought.

The purpose of the present work is to evaluate the level of sophistication necessary to model the turbulent heat fluxes, when the dynamic field is modeled by the Elliptic-Blending Reynolds-Stress Model (EB-RSM, Manceau & Hanjalić, 2002). A particular attention will be paid to the reproduction of the selective damping of the wall-normal fluctuations due to wall blocking by the introduction of the elliptic blending strategy into the heat flux model, as proposed by Shin \textit{et al.} (2008).

2 Elliptic Blending Reynolds-Stress Model

In order to replicate in a RSM the blocking of the wall-normal fluctuation, following the pioneering work of Durbin (1993) on elliptical relaxation, the EB-RSM, proposed by Manceau & Hanjalić (2002) and modified by Manceau (2005), is formulated in order to satisfy the near-wall balance

\[ \phi_{ij}^* - \varepsilon_{ij} = -D_{ij}, \]  

where \( \phi_{ij}^* \) and \( D_{ij} \) denote velocity-pressure-gradient correlation, dissipation and molecular diffusion tensors, respectively. This constraint is fulfilled by using a blending of standard models for \( \phi_{ij}^* - \varepsilon_{ij} \) (hereafter the SSG model, Speziale \textit{et al.}, 1991, and the isotropic model \( \varepsilon_{ij}^* = 2/\beta \varepsilon_{ij} \), valid far from the wall, and near-wall models for \( \phi_{ij}^* - \varepsilon_{ij} \) satisfying the asymptotic behaviour in the near-wall region

\[ \phi_{ij}^* - \varepsilon_{ij} = (1 - \alpha^3)(\psi_{ij}^* - \varepsilon_{ij}^*) + \alpha^3(\phi_{ij}^* - \psi_{ij}^*). \]  

In order to preserve the nonlocal character of the blocking effect, similarly to Durbin’s elliptical relaxation model, the blending function \( \alpha \) is obtained by the elliptical relaxation equation

\[ \alpha = -L^2 \nabla^2 \alpha = 1 \quad \text{with} \quad \alpha|_w = 0. \]  

and goes from 0 at the wall to 1 far from the wall. If the simple model \( \psi_{ij}^* = \omega_{ij}/k \varepsilon \) is used, the asymptotic analysis in the vicinity of the wall shows that \( \phi_{ij}^* \) must be modelled by

\[ \phi_{ij}^* = -\delta \frac{\varepsilon}{k} \left[ \frac{\varepsilon_{ij}^*}{\varepsilon} \right] \frac{1}{2} \left( \nabla \alpha \cdot \nabla \parallel \alpha \right) \right], \]

where \( \parallel \alpha / \parallel \nabla \alpha \parallel \) is a unit vector providing the wall-normal direction.

However, one can wonder if the EB-RSM is still valid when buoyancy is introduced, i.e., if buoyancy modifies the near-wall balance (1). Close to a wall located at \( y = 0 \), Taylor-series expansions lead to the behaviours given in table 1 for the Reynolds stress budgets. It can be seen that the production due to buoyancy \( G_{ij} \) does not affect the balance (1), and, in particular, that \( G_{ij\delta}/\phi_{ij\delta} = O(y^{-1}) \) for all \( \gamma \) and \( \delta \). It can be concluded that the EB-RSM does not require any further modification than introducing buoyancy terms. However, one could expect a modification of the transition from the near-wall behaviour to the far-from-the-wall behaviour, which would question the validity of Eq. (3). However, using the DNS data of Kasagi & Nishimura (1997), in the mixed convection case of a vertical channel with a pressure gradient and a temperature difference between the two walls, it
buoyancy, the pressure term, the dissipation, the turbulent and molecular diffusion, respectively.

Algebraic forms of the turbulent heat flux transport equation can be derived, similarly to the Reynolds stress transport equations (see, for example, Ocení et al., 2008). Indeed, the transport equations for the non-dimensional heat flux vector, $\bar{\varepsilon}_i = \bar{w}_i \bar{\theta}/(\sqrt{k} \sqrt{\bar{\theta}^2})$, can be easily derived from Eq. (5). Again, similarly to what is usually assumed for the anisotropy tensor (Rodi, 1976), weak equilibrium hypotheses can be used

$$\frac{d\bar{\varepsilon}_i}{dt} = 0 = 1 \frac{1}{u_i \bar{\theta}} \frac{D_{\bar{\varepsilon}_i}}{1} = 1 = \frac{1}{ \bar{\varepsilon} \bar{\theta}} = \frac{1}{ \bar{\varepsilon} \bar{\theta}}$$

where $D_{\bar{\varepsilon}_i}$, $D_{\bar{\theta}}$ and $D_{\bar{\theta}^2}$ denote the total diffusion of $u_i \bar{\theta}$, $k$ and $\bar{\theta}^2$, respectively. These hypotheses lead to the algebraic equation for the heat flux

$$(P_{\bar{\varepsilon}_i} - \frac{\beta u_i}{k} H_{\bar{\varepsilon}_i} - \frac{\beta u_i}{k} H_{\bar{\theta}^2}) + \phi_{\bar{\varepsilon}_i} - (\bar{\varepsilon}_i - \frac{\bar{\varepsilon}}{k} - \bar{\varepsilon} \bar{\theta}) = 0$$

The most general model for the pressure scrambling term $\phi_{\bar{\varepsilon}_i} = -\rho \bar{\theta} / \partial y / \partial x_3$ used in the present paper consists in the combination of the isotropization of production model of Lauder (1975) for the rapid part and the nonlinear model of Kenjereš et al. (2005) for the slow part

$$\phi_{\bar{\varepsilon}_i} = -C_{\bar{\varepsilon}_i} \bar{\varepsilon} \bar{\theta} = 0 = 1 = 0$$

The dissipation of the heat flux is assumed isotropic: $\bar{\varepsilon}_i = 0$. Introducing this model in Eq. (7), and assuming that the production and the dissipation terms of both $k$ and $\bar{\theta}^2$ are locally in balance (i.e., turbulence equilibrium, for details see Hanjalić, 2002), Eq. (5) reduces to the Non-Linear Algebraic Flux Model (NLAFM) (Kenjereš et al., 2005)

$$\bar{u}_i \bar{\theta} = -C_{\bar{\theta}} \bar{\varepsilon} \bar{\theta} = 0 = 1$$

where $C_{\bar{\theta}} = C_{\bar{\theta}}/C_{\bar{\varepsilon}_1}$, $C_{\bar{\varepsilon}_1} = 1 - C_{\bar{\varepsilon}_2}$, $C_{\bar{\varepsilon}_2} = 1 - C_{\bar{\varepsilon}_3}$, $C_{\bar{\varepsilon}_3} = 1 - C_{\bar{\varepsilon}_4}$, $\eta = 1 - C_{\bar{\varepsilon}_4}$ and $\chi = C_{\bar{\varepsilon}_1} C_{\bar{\varepsilon}_2} C_{\bar{\varepsilon}_3}$. Two simplifications of this model will be used in the following: the linear Algebraic Flux Model (AFM, $\chi = 0$) and the Generalized Gradient Diffusion Hypothesis (GDDH, $\xi = 1 - \eta = \chi = 0$). Throughout the present paper, the coefficients that are generally recommended (Hanjalić, 2002), $C_{\bar{\varepsilon}_1} = 3.0$, $C_{\bar{\varepsilon}_2} = 0$, $C_{\bar{\varepsilon}_3} = 0.55$, will be used. Note that the coefficient $C_{\bar{\theta}}$, which should be equal to $1/C_{\bar{\varepsilon}_1}$, has been modified by the introduction of the coefficient $C_{\bar{\theta}}^\prime$; indeed, the assumptions used in the algebraic methodology (weak and turbulence equilibrium) make necessary the introduction of this recalibration coefficient. The value generally admitted for this coefficient (Hanjalić, 2002) is about $C_{\bar{\theta}}^\prime = 0.6$. 

<table>
<thead>
<tr>
<th>$D_{ij}^\prime$</th>
<th>$\phi_{ij}^\prime$</th>
<th>$\varepsilon_{ij}$</th>
<th>$P_{ij}$</th>
<th>$D_{ij}^\bar{\varepsilon}$</th>
<th>$D_{ij}$</th>
<th>$G_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\theta}^2$</td>
<td>$\bar{\theta}(y)$</td>
<td>$\bar{\theta}(y^2)$</td>
<td>$\bar{\theta}(y^2)$</td>
<td>$\bar{\theta}(y^2)$</td>
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<td>$\bar{\theta}(y^3)$</td>
<td>$\bar{\theta}(y^3)$</td>
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<td>$\bar{\theta}^2$</td>
<td>$\bar{\theta}(y^4)$</td>
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<td>$\bar{\theta}(y^5)$</td>
<td>$\bar{\theta}(y^5)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Asymptotic behaviours of the terms of the Reynolds stress budget in buoyancy affected flows.
4 Elliptic Blending-Algebraic Flux Model

The model (8) and, consequently, the AFM, does not account for the effect of the wall on turbulence. In order to overcome this limitation, the main proposal of the present paper is to apply the elliptic blending strategy (see section 2) to the heat fluxes in order to derive the Elliptic Blending-Algebraic Flux Model (EB-AFM). This model is actually the algebraic version of the elliptic-blending heat-flux transport model recently proposed by Shin et al. (2008).

Similarly to the EB-RSM, the main idea underlying the derivation of the model of Shin et al. (2008) is that it must satisfy the correct asymptotic balance in the near-wall limit

$$\phi^*_{i1} - \varepsilon_{i1} = -D^\omega_{i1},$$

(10)

which can be achieved by using the same type of blending formula as in the EB-RSM

$$\phi^*_{i1} - \varepsilon_{i1} = (1 - \alpha^3)(\phi^*_{i1} - \varepsilon_{i1}) + \alpha^3(\phi^h_{i1} - \varepsilon^h_{i1}).$$

(11)

The model for the far-from-the-wall part $\phi^h_{i1} - \varepsilon^h_{i1}$ is the one described in the previous section. It can be shown (Shin et al., 2008) that the balance (10) can be satisfied by using

$$\varepsilon^w_{i1} = C_\varepsilon \frac{\varepsilon}{k} u_i \theta, \text{ where } C_\varepsilon = \frac{1}{2} \left( 1 + \frac{1}{Pr} \right)$$

(12)

and

$$\phi^w_{i1} = - \left[ 1 + \frac{1}{2} \left( 1 + \frac{1}{Pr} \right) \right] \frac{\varepsilon}{k} u_i \theta n_i n_j.$$

(13)

The same derivation as in the previous section leads to the EB-AFM

$$\overline{u_i \theta} = -C_{\theta} \frac{k}{\varepsilon} \left[ \xi u_i u_j \frac{\partial T}{\partial x_j} + \xi u_i \theta \frac{\partial u_j}{\partial x_j} + \eta \beta \frac{\theta^2}{2} + \gamma \frac{\varepsilon}{k} u_i \theta n_i n_k - \chi k a_{ij} u_j \theta \right],$$

(14)

which differs from the model (9) by the additional term $\gamma \frac{\varepsilon}{k} u_i \theta n_i n_k$, which sensitizes the model to the orientation of the wall, and, above all, by the fact that the coefficients are now dependent on the blending function $\alpha$:

$$C_\theta = \frac{C_{\theta}^0}{\alpha^3 C_{\theta1} + (1 - \alpha^3) C_{\varepsilon}}.$$

(15)

$$\zeta = 1 - \alpha^2 C_{\theta2}; \chi = \alpha^3 C_{\theta1} C_{\varepsilon} / C_{\theta1}; \xi = 1 - \alpha^3 C_{\theta2}; \eta = 1 - \alpha^3 C_{\theta3}; \gamma = (1 - \alpha^3) \left( 1 + (1 + 1/Pr)/2 \right).$$

In regions far from the wall ($\alpha \to 1$), the influence of the elliptic blending method vanishes and model (9) is recovered. Note that the elliptic blending method does not introduce any new coefficient to calibrate.

5 Evaluation of the new model

Selection of test cases

In order to cover a wide spectrum of convection regimes, four different 1D test cases are selected. The geometry is the same for all the cases and consists of 2 infinite plane walls located at $y = 0$ and $y = 2h$. The flow is generated by the superposition of an imposed pressure gradient in the $x$-direction and either a temperature difference $T_0 - T_2h = \Delta T$ or a constant wall heat flux $\dot{q}_w$. The gravity is oriented either along the $x$-axis (vertical channel) or along the $y$-axis (horizontal channel). The flow is completely characterized by four parameters: the friction Reynolds number $Re_x = h u_x / \nu$ (averaged on the two walls), the Grashof number $Gr = \beta g \Delta T(2h)^3 / \nu^2$, the orientation of gravity $\gamma / g$ and the Prandtl number $Pr = \nu / \alpha$. Table 2 provides these parameters for each of the four test cases, as well as the authors of the corresponding DNS databases.

A priori tests

The NL-AFM was proposed by Kenjereš et al. (2005) and calibrated by a priori tests in natural convection. Fig. 2 shows a comparison of a priori results given by the AFM and the NL-AFM. It can be indeed seen that the nonlinear model improves the results in natural convection. However, both the predictions of $\overline{u_i \theta}$ and $\overline{v \theta}$ are completely wrong in the mixed convection case: actually, the sign of the coefficient $C_{\theta1}$ must be changed in order to obtain heat fluxes with the correct sign. These results lead us to get rid of the nonlinear term by using $C_{\theta1} = 0$ hereafter.

Fig. 4 shows a priori results of the GGDH, the AFM and the EB-AFM for the four test cases. It is clearly seen that the use of elliptic blending has a very beneficial effect on the prediction of the tangential heat flux $\overline{u_i \theta}$. The improvement of this component is due to the introduction of variable coefficients, and, in particular, $C_{\theta}$. Indeed, the denominator of Eq. (15) varies from $C_{\theta1} = 3.0$ far from the wall to $C_{\varepsilon} = (1 + 1/Pr) / 2 = 1.20$ at the wall.

One can wonder why such an effect is not visible on $\overline{v \theta}$ as well: in particular, for the 3 cases for which the channel is vertical, the wall-normal component $\overline{v \theta}$ is only marginally modified. This is due to the presence in Eq. (14) of the additional term $\gamma k a_{ij} u_j \theta n_i n_k$: indeed, the $\overline{v \theta}$ component can be written

$$\overline{v \theta} = -C_\theta \frac{k}{\varepsilon} \frac{\partial \theta}{\partial y},$$

(16)

where

$$C_\theta = \frac{1}{\alpha^3 C_{\theta1} + (1 - \alpha^3) (2 + 1/Pr)}.$$

(17)

For $Pr = 0.71$, the parameter $(2 + 1/Pr)$ is equal to 3.41, which is rather close to $C_{\theta1} = 3.0$, such that $C_\theta$ is almost constant and $\overline{v \theta}$ is only weakly modified compared to the GGDH and the AFM. This coincidence is actually the reason why the GGDH and the AFM are able to reproduce correctly the $\overline{v \theta}$ component throughout the channel, although they do not account for wall effects. The situation would be very different for other values of the Prandtl number: for example, for $Pr = 0.025$ (Kawamura et al., 1999) the parameter $(2 + 1/Pr)$ is equal to 41.
In order to confirm by full computations this favourable behaviour of the EB-AFM in a priori tests, the finite volume code Saturne (Archambeau et al., 2004), developed at EDF, is used, as well as a dedicated 1D, finite difference code for cross-checking.

All the equations of the models are identical, except, of course, for the heat flux model. The dissipation of $\theta^2$ is obtained from

$$\varepsilon_{\theta^2} = \frac{\tau_{\theta}}{R} \frac{\varepsilon}{k}$$

(18)

It is usual (Hanjalić, 2002) to consider that the time-scale ratio $R$ can be assumed constant and equal to 0.5. However, it is easy to show, from an asymptotic analysis in the vicinity of the wall, that its wall value is $R = Pr$. Therefore, the second proposal of the present paper is to approximate the time-scale ratio by

$$R = (1 - \alpha^3) Pr + \alpha^3 0.5.$$  

(19)

Fig. 3 illustrates the advantage of using this new approximation.

In order to fairly compare the different models, they are calibrated using the same procedure: the standard coefficients of the pressure scrambling term $\phi_\theta i$ are kept unchanged: $C_{\theta 1} = 3.0$, $C_{\theta 2} = 0$, $C_{\theta 3} = C_{\theta 3} = 0.55$. The coefficient $C_{\theta}^0$ is calibrated such that the mean temperature profile in the case of forced convection is correctly reproduced. This procedure leads to $C_{\theta}^0 = 0.705$ for the GGDH and the AFM, and to $C_{\theta}^0 = 0.68$ for the EB-AFM.

In the present section, the possibility of using an Elliptic-Blending GGDH (EB-GGDH), i.e., to use Eq. (14) with $\xi = \eta = \chi = 0$, is also investigated.

Figs. 5, 6 and 7 show the results obtained with the GGDH, the AFM, the EB-AFM and the EB-GGDH for the four test cases. The heat fluxes globally conform well to the a priori tests, except for discrepancies due a misreproduction of the mean profiles. The EB-GGDH model gives results very close to those of the EB-AFM, except in the natural convection case, for which the influence of the buoyancy term in Eq. (14) is significant. Therefore, the EB-GGDH appears as a very acceptable simplified model for mixed and forced convection.

It clearly appears that the mean velocity and temperature profiles are weakly influenced by the heat flux model. This is explained by the fact, emphasized at the end of the previous section, that $\overline{\theta v}$ is only marginally modified by the introduction of elliptic blending procedure.

The main discrepancies with the DNS appear in the unstable stratification case and the natural convection case. In the former case, the analysis of the heat flux budget (not shown here) shows that this can be related to the misrepresentation of the turbulent diffusion term at the center of the channel, due to the weak equilibrium hypothesis. In the natural convection case, which is directly driven by buoyancy, discrepancies in the heat flux prediction are amplified by the strong

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**Table 2: Description of the test cases.**

<table>
<thead>
<tr>
<th>Test Case</th>
<th>Re$_x$</th>
<th>Gr</th>
<th>$g/g$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forced convection</td>
<td>636</td>
<td>0</td>
<td>NA</td>
<td>0.71</td>
</tr>
<tr>
<td>Kasagi et al. (1994)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mixed convection</td>
<td>150</td>
<td>$9.6 \times 10^5$</td>
<td>$-\phi_\theta$</td>
<td>0.71</td>
</tr>
<tr>
<td>Kasagi &amp; Nishimura (1997)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unstable stratification</td>
<td>150</td>
<td>$1.3 \times 10^6$</td>
<td>$-\phi_\theta$</td>
<td>0.71</td>
</tr>
<tr>
<td>Iida &amp; Kasagi (1995)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Natural convection</td>
<td>0</td>
<td>$5 \times 10^5$</td>
<td>$-\phi_\theta$</td>
<td>0.709</td>
</tr>
<tr>
<td>Versteeg &amp; Nieuwstadt (1998)</td>
<td></td>
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</tbody>
</table>

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**Figure 2:** A priori test of the Nonlinear AFM (NL-AFM) in natural and mixed convection.

**Figure 3:** Computations: comparison of the new model for the time-scale ratio $R$ with DNS in the mixed convection case.
coupling between the dynamic and the thermal fields. Consequently, the Nusselt number is overestimated by a factor of 2, contrary to the other cases, where it is correctly reproduced.

6 Conclusions

The introduction of the elliptic blending method in algebraic models for the turbulent heat fluxes is straightforward and only consists in an additional term and variable coefficients. The resulting model, the EB-AFM, significantly improves the prediction of the wall tangential heat flux in four different channel flows, covering the range of natural, mixed and forced convection, both in a priori tests and in full computations.

A simplified version, the EB-GGDH, which does not require the resolution of a transport equation for $\overline{\theta^2}$, is also proposed, and appears to be a sufficient model for mixed and forced convection.

These models are to be tested in more complex geometries, but they appear as very promising for industrial applications.

References


Figure 4: A priori tests