LAMINAR NATURAL CONVECTION OF BINGHAM FLUIDS IN A TRAPEZOIDAL ENCLOSURE HEATED FROM THE BOTTOM

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ABSTRACT

Laminar, steady-state, natural convection of Bingham fluids in trapezoidal enclosures with a heated bottom wall, adiabatic top wall and cooled inclined sidewalls has been analysed based on numerical simulations for a range of different values of nominal Rayleigh numbers (i.e. $10^3 \leq Ra \leq 10^5$), Bingham numbers (i.e. $0 \leq Bn \leq 0.5$), and inclination angles (i.e. $30° \leq \varphi \leq 60°$) for a representative nominal Prandtl number (i.e. $Pr = 10^3$). To conduct the parametric investigation, a commercial finite-volume solver has been used. It has been found that the mean Nusselt number $\overline{Nu}$ increases with increasing Ra due to the strengthening of advective transport. However, an increase in the sidewall inclination φ leads to a decrease in $\overline{Nu}$. The value of $\overline{Nu}$ was found to decrease with increasing Bn. At high values of Bn, fluid flow essentially ceases within the enclosure and the heat transfer takes place predominantly due to conduction and, therefore, the value of $\overline{Nu}$ settles to a constant value, for a given value of φ, irrespective of the value of nominal Ra. Furthermore, an expression for the mean Nusselt number $\overline{Nu}$ in a trapezoidal enclosure with heat bottom wall, cooled inclined sidewalls, and an adiabatic top wall accounting for the considered range of Rayleigh numbers Ra, Bingham numbers Bn and inclined wall angles φ has been identified which provides adequate approximation of the corresponding value obtained from the simulation.

1. INTRODUCTION

A yield stress fluid is a special type of non-Newtonian fluid which acts as a solid below a threshold stress (i.e., a yield stress τ_y) but flows like a fluid above this critical stress [1]. The use of yield stress fluids in industrial applications is wide-ranging with applications in nuclear waste cooling, food and chemical processing as well as cryogenic storage. The rheological behaviour of yield stress fluids is often modelled by a Bingham model which provides a linear strain rate dependence of the shear stress. The analysis of heat transfer in Bingham fluids is, therefore, one of practical interest but also of academic interest. The study of Bingham fluids has been considered for square enclosures [2], offering correlations for the mean Nusselt number $\overline{Nu}$. However, relatively limited attention has been directed to the natural convection of yield stress fluids in non-rectangular enclosures. The Rayleigh–Bénard convection (i.e., heated bottom wall and cooled top wall with adiabatic inclined side walls) within trapezoidal enclosures filled with viscoplastic fluid has been analysed by Aghighi et al. [3] across a range of parameters (i.e. the angle of inclination of the side walls φ, Rayleigh number Ra and Prandtl number Pr). However, to the best of the authors' knowledge, the natural convection in Bingham fluids in a trapezoidal enclosure with heat bottom wall, cooled inclined sidewalls and an adiabatic top wall is yet to be considered in detail. Therefore, the objectives of the current study are, as follows:

1. To investigate the influence of the geometry of a trapezoidal cavity, Rayleigh number Ra and Bingham number Bn on the natural convection behaviour in Bingham fluids in a trapezoidal enclosure with heat bottom wall, cooled inclined sidewalls, and an adiabatic top wall.

2. To identify an expression for the mean Nusselt number $\overline{Nu}$ in the considered configuration across the range of Ra, Bn and φ examined.

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2. MATHEMATICAL BACKGROUND & NUMERICAL IMPLEMENTATION

The schematic of the configuration considered in the current analysis is provided in Fig. 1a where L is the length of the bottom heated wall, H is the height of the trapezium and φ is the angle of inclination of the sidewall. The heated bottom wall is kept at a temperature T_H and the two inclined sidewalls are kept at a temperature T_C. It is assumed that $T_H > T_C$. The top wall is considered to be adiabatic in nature. The no-slip condition is applied to all walls. The flow is assumed to be laminar, incompressible, steady, and two-dimensional (i.e., the physical flow domain is assumed to be an infinitely long channel and, therefore, the third dimension is considered not to affect the flow field). The conservation equations for mass, momentum, and energy take the following form for the current study:

$$\partial u_i/\partial x_i = 0 \tag{1i}$$

$$\rho u_j\left(\partial u_i/\partial x_j\right) = -(\partial p/\partial x_i) + \delta_{2i}\rho g\beta(T_H - T_C) + \partial \tau_{ij}/\partial x_j \tag{1ii}$$

$$\rho u_j C\left(\partial T/\partial x_j\right) = k\left(\partial^2 T/\partial x_j \partial x_j\right) \tag{1iii}$$

where u_i (x_i) is the i^{th} component of velocity (spatial coordinate), ρ is the density, p is the pressure, g is the acceleration due to gravity, β is the thermal expansion coefficient, τ_{ij} is the stress tensor, C is the specific heat, T is the temperature and k is the thermal conductivity. In Eq. 1ii, the Kronecker delta δ_{2i} is employed to ensure that the buoyancy effect only occurs in the vertical direction (i.e., x_2 direction). The Bingham model for yield stress fluids can be expressed as [2]:

$$\underline{\underline{\dot{\gamma}}} = 0 \qquad \text{for } \tau \le \tau_y \tag{2i}$$

$$\underline{\underline{\tau}} = \left(\mu + \tau_y/\dot{\gamma}\right)\dot{\gamma}_{ij} \qquad \text{for } \tau > \tau_y \tag{2ii}$$

where the components of the strain rate tensor $\dot{\gamma}$ are given by: $\dot{\gamma}_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$. In Eq. 2, $\tau = \left[0.5\left(\underline{\underline{\tau}}:\underline{\underline{\tau}}\right)\right]^{0.5}$ and $\dot{\gamma} = \left[0.5\left(\underline{\underline{\dot{\gamma}}}:\underline{\underline{\dot{\gamma}}}\right)\right]^{0.5}$. The stress-shear rate characteristics of a Bingham fluid is approximated here by the bi-viscosity regularisation [4]:

$$\underline{\underline{\tau}} = \mu_{yield}\underline{\underline{\dot{\gamma}}} \qquad \text{for } \dot{\gamma} \le \tau_y/\mu_{yield} \tag{3i}$$

$$\underline{\underline{\tau}} = \tau_y(\underline{\underline{\dot{\gamma}}}/\dot{\gamma}) + \mu\underline{\underline{\dot{\gamma}}} \qquad \text{for } \dot{\gamma} > \tau_y/\mu_{yield} \tag{3ii}$$

where μ_{yield} is the yield viscosity and μ is the plastic viscosity such that the solid material is represented by a high viscosity fluid [4]. A value of $\mu_{yield} \ge 1000\mu$ satisfactorily mimics the true Bingham model according to its proponents [4], and here $\mu_{yield}/\mu = 10^4$ is chosen to ensure the high-fidelity of the computational results. According to Buckingham's pi theorem, the Nusselt number Nu (defined as $Nu = hL/k$ where $h = q_w/(T_H - T_C)$ is the local heat transfer coefficient where q_w is the wall heat flux at the bottom hot wall) can be expressed in this configuration as $Nu = f(Ra, Pr, H/L, \varphi, Bn)$ where the Bingham number Bn, Rayleigh number Ra, and Prandtl number Pr are defined as $Ra = \rho g\beta\Delta TL^3/(\mu\alpha)$, $Bn = \tau_y L/\left(\mu\sqrt{g\beta\Delta TL}\right)$ and $Pr = C\mu/k$ where $\Delta T = (T_H - T_C)$, and $\alpha = k/\rho C$ is the thermal diffusivity. The present analysis considers the aspect ratio H/L to be unity (i.e., $H/L = 1.0$).

A finite-volume (i.e., ANSYS-FLUENT) solver [5] has been used for solving the governing equations. A second-order upwind scheme (second-order central difference) has been used for the discretisation of convective (diffusive) terms. The coupling of velocity and pressure components is achieved using the SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm [5], where the convergence criteria were set to 10^{-6} for all relative (scaled) residuals. The boundary

conditions are: $T = T_H$, $u_1 = u_2 = 0$ at the bottom wall; $T = T_C$, $u_1 = u_2 = 0$ at the sidewalls and $\partial T/\partial y = 0$, $u_1 = u_2 = 0$ at the top wall. The parameters considered in the current study are: $Ra = 10^3, 10^4, 10^5$; $0 \leq Bn \leq 0.5$; and $\varphi = 30^\circ, 45^\circ, 60^\circ$. The Prandtl number $Pr = 10^3$ was considered for all cases. A mesh independence analysis has been completed and a non-uniform unstructured triangular mesh of 22500 cells (shown in Fig. 1b) is used for the study. This mesh provides agreement of Nu on the hot wall to within 2% with a mesh of 30625 cells but with a 26% reduction in computational cost, giving balance between accuracy and cost for the parametric investigation where 80 simulations were considered. Figure 1b also provides the non-dimensional temperature $\theta = (T - T_C)/(T_H - T_C)$ field of an example case (i.e., $Ra = 10^3$, $Bn = 0.5$, $\varphi = 30°$). Furthermore, the currently considered numerical implementations have been tested against benchmarks with the natural convection of Newtonian fluids in a square enclosure (i.e., $\varphi = 0^\circ$) with differentially heated sides [2] and natural convection heat transfer in partially divided trapezoidal cavities [6]. For both benchmark studies, satisfactory agreements (i.e., typically within 0.5% but, at most, 2% across all benchmark cases considered) were obtained.

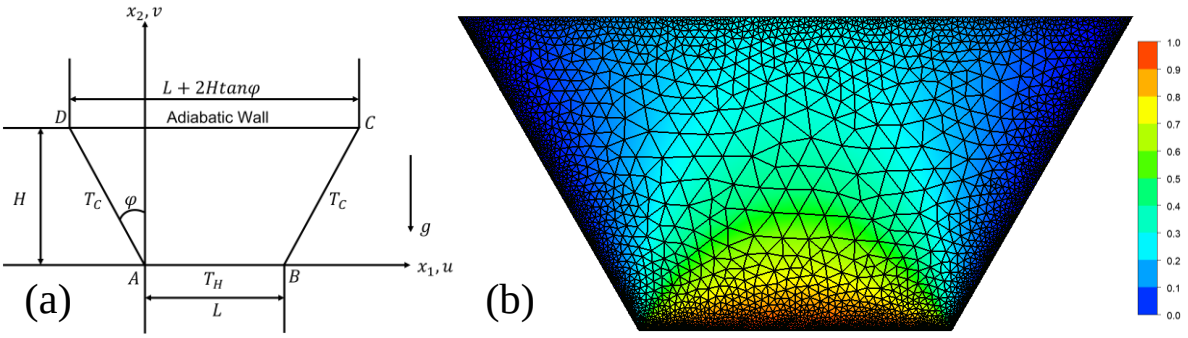

Figure 1: (a) Schematic of considered configuration, and (b) the non-dimensional temperature $\theta = (T - T_C)/(T_H - T_C)$ field for the $Ra = 10^3$, $Bn = 0.5$, $\varphi = 30°$ case with the mesh superimposed.

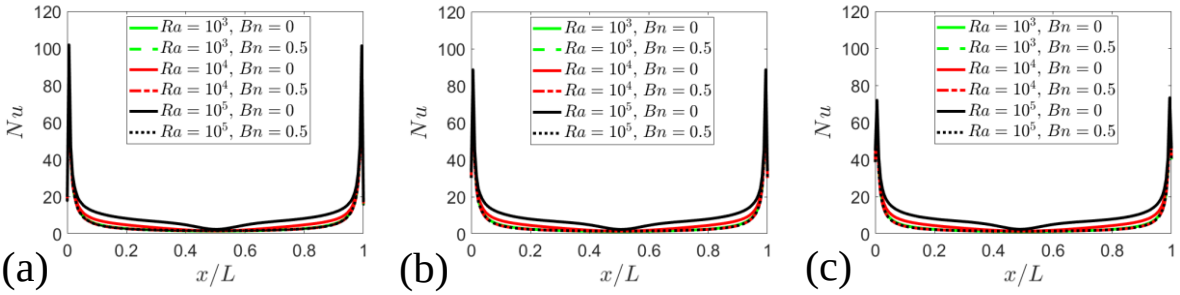

Figure 2: The variations of local Nusselt number Nu on the hot bottom wall with normalised horizontal distance x_1/L for (a) $Ra = 10^3$, $Ra = 10^4$ and $Ra = 10^5$ where $Bn = 0.5$, $Pr = 10^3$ compared to the corresponding Newtonian fluid for (a) $\varphi = 30°$, (b) $\varphi = 45°$ and (c) $\varphi = 60°$ configurations.

3. RESULTS & DISCUSSION

In the following sections, the effects of Ra, Bn, and φ on the heat transfer behaviour in the trapezoidal enclosure are discussed. The variations of the local Nusselt number Nu on the hot wall with normalised horizontal distance x_1/L for $Ra = 10^3$, 10^4 and 10^5 where $Bn = 0.5$, $Pr = 10^3$, in comparison to the corresponding Newtonian fluid (i.e. $Bn = 0$ where the yield stress $\tau_y = 0$), are shown for $\varphi = 30°$, $\varphi = 45°$, and $\varphi = 60°$ in Figs. 2a-c, respectively. Figures 2a-c show that Nu increases with increasing Ra for both the Bingham and Newtonian fluids considered. Furthermore, Figs. 2a-c show that the values of Nu are generally greater for the Newtonian fluid than the Bingham fluid across x_1/L with the same nominal Ra. This difference is most apparent in $Ra = 10^5$ cases and is due to the

strengthening of buoyancy effects with increasing Ra which will have the greatest effect in the $Bn = 0$ cases where there is yield stress $\tau_y = 0$. Figures 3a-f show the contours of non-dimensional temperature θ for Bingham fluids of $Bn = 0.5$ (i.e., top row) and the corresponding Newtonian fluid (i.e., bottom row) for $Ra = 10^3$, 10^4 and 10^5 where $Pr = 10^3$ and $\varphi = 30°$. Figures 3d-f show that there are significant changes in the behaviour of θ with increasing Ra for Newtonian fluids. However, Figs. 3a-c show that for Bingham fluids the contours of θ show negligible variation with increasing Ra for $Bn = 0.5$. This suggests that for sufficiently large values of Bn, conduction begins to play the dominant role in thermal transport and the nominal Rayleigh number Ra no longer affects the value of Nu.

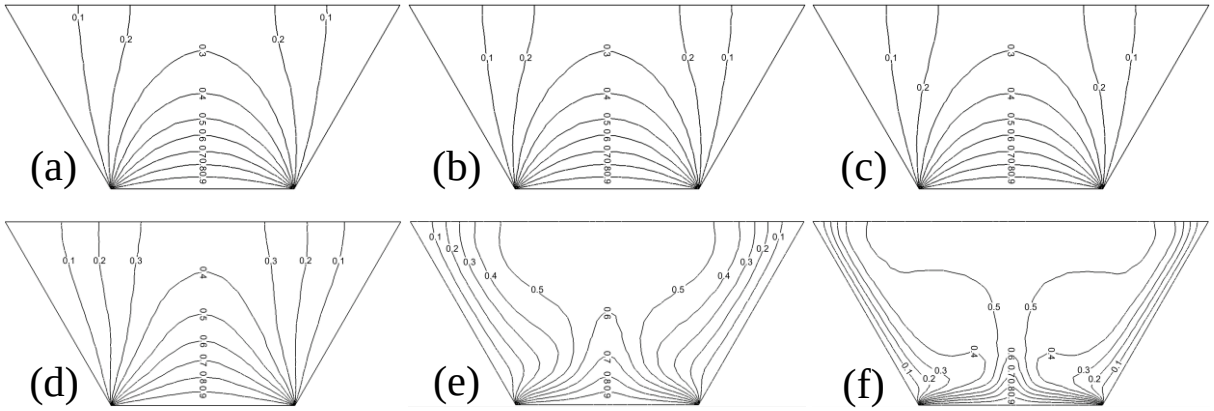

Figure 3: Contours of non-dimensional temperature θ for $Bn = 0.5$ (i.e., top row) and the corresponding Newtonian fluid (i.e., bottom row) for $Ra = 10^3$, 10^4 and 10^5 where $Pr = 10^3$ and $\varphi = 30°$.

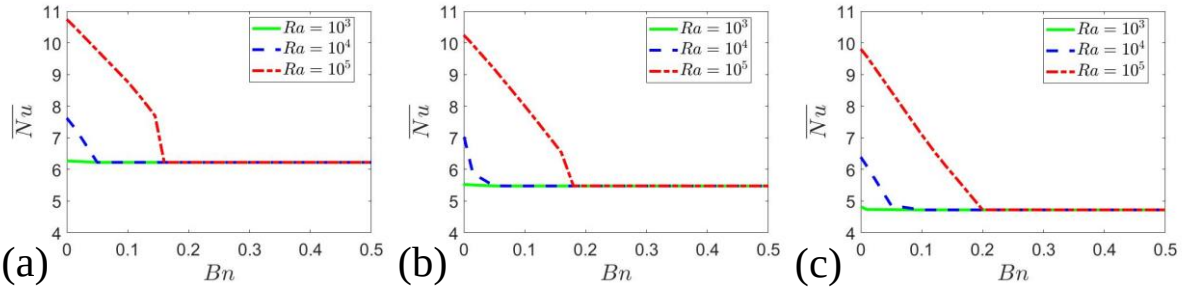

Figure 4: Variations of the mean Nusselt number $\overline{Nu}$ on the hot bottom wall with Bingham number Bn for $Ra = 10^3$, 10^4 and 10^5 where $Pr = 10^3$ for (a) $\varphi = 30°$, (b) $\varphi = 45°$, and (c) $\varphi = 60°$.

The effects of Bn on the nature of the heat and mass transfer in the trapezoidal cavity can further be shown through the variation of the mean Nusselt number $\overline{Nu}$ with Bn as shown for $Ra = 10^3$, 10^4 and 10^5 where $Pr = 10^3$ for $\varphi = 30°$, $45°$ and $60°$ in Figs. 4a-c, respectively. Figures 4a-c show that, for a given set of values of Ra and φ, $\overline{Nu}$ is found to decrease as Bn increases until $\overline{Nu}$ plateaus to a constant value (i.e., at $Bn = Bn_{max}$) which, for a given φ, is common across all Ra considered. For larger values of Bn, where τ_y is sufficiently large relative to the viscous stress, despite increases in Ra, no significant flow is induced within the enclosure and conduction heat transfer plays the dominant role as previously discussed. Importantly, however, Figs. 4a-c shows that an increase in Ra leads to an increase in $\overline{Nu}$ for sufficiently low values of Bn where flow is induced, and convection heat transfer occurs. Moreover, for higher values of Ra where buoyancy forces are greater, the value of Bn at which $\overline{Nu}$ settles to a constant value also increases. The effect of Bn on the flow behaviour within the enclosure can further be examined by considering the non-dimensional vertical velocity $U_2 = u_2 L/\alpha$ at the vertical centreline (vertical line of symmetry) as shown for $Ra = 10^5$ where $Pr = 10^3$ for $\varphi = 30°$, in Figs. 5a-c, respectively. Figures 5a-c show that U_2 decreases with increasing Bn. This corroborates the observations from Figs. 4a-c which indicates that an increase in Bn suggests strengthening of flow resistance relative

to the buoyancy forces and it is reflected in a reduction in the non-dimensional vertical velocity U_2. As such, this suggests that a further increase in Bn will eventually lead to a negligible value of U_2 where conduction will become the sole heat transfer mechanism.

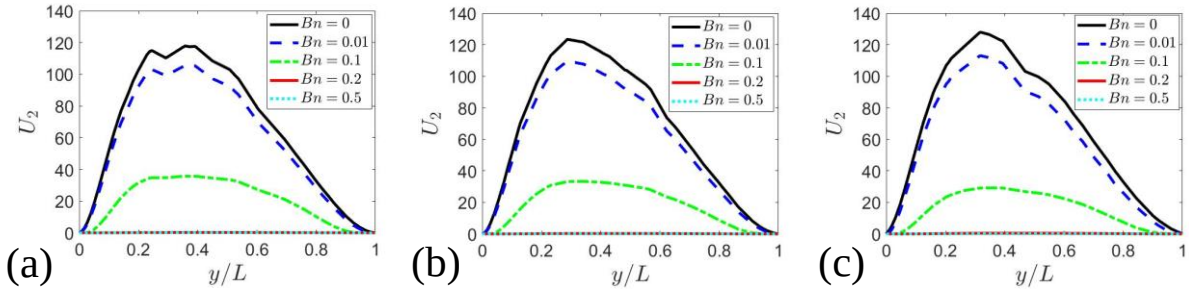

Figure 5: Variation of non-dimensional vertical velocity $U_2 = u_2 L/\alpha$ along the vertical centreline for different Bingham numbers for $Ra = 10^5$ and $Pr = 10^3$ for (a) $\varphi = 30°$, (a) $\varphi = 45°$, and (c) $\varphi = 60°$.

The effect of φ on the heat transfer behaviour can be gleaned by considering the variation of $\overline{Nu}$ with Bn for $\varphi = 30°$, $45°$ and $60°$, which is shown in Fig. 6. It is evident from Fig. 6 that an increase in the vertical angle φ leads to a decrease in $\overline{Nu}$ which is due to the walls at temperature T_C (i.e., cold walls, inclined to the vertical) becoming longer leading to greater area for losing heat from the cavity and, therefore, a smaller amount of heat flux is needed for higher values of φ to maintain the same temperature difference $\Delta T = (T_H - T_C)$. The observed effects of Ra, Bn and φ on the heat transfer behaviour must be accounted for when obtaining an expression for $\overline{Nu}$.

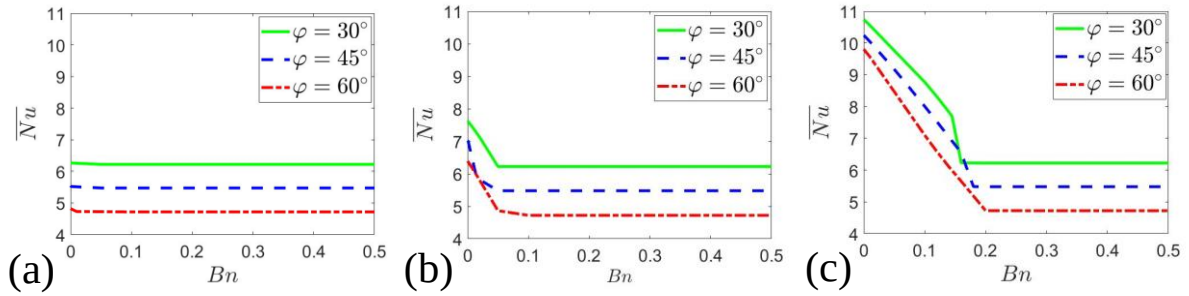

Figure 6: The variation of mean Nusselt number $\overline{Nu}$ for the hot bottom wall with Bingham number Bn for $\varphi = 30°$, $45°$ and $60°$ where $Pr = 10^3$ for (a) $Ra = 10^3$, (b) $Ra = 10^4$, and (c) $Ra = 10^5$.

Previous analyses have developed expressions for the mean Nusselt number $\overline{Nu}$ for Bingham fluids in different enclosures across a range of Ra, Pr, Bn and, where appropriate, aspect ratios which have extended upon the expressions for Newtonian fluids [2,6,7]. Based on scaling arguments [6,7], an expression can be proposed which varies in the region of $0 \le Bn \le Bn_{max}$ accounting for the fall in $\overline{Nu}$ in this range and takes a constant value for $Bn > Bn_{max}$. As such, the following expression for $\overline{Nu}$ can be proposed which extends expression for square enclosures for trapezoidal enclosures:

$$\overline{Nu} = 1 + \frac{ARa^{1/2}}{\left[\frac{Bn}{2} + \frac{1}{2}\sqrt{Bn^2 + 4\left(\frac{Ra}{Pr}\right)^{\frac{1}{2}}}\right]}\left[1 - \left(\frac{Bn}{Bn_{max}}\right)^{3/5}\right]^b \text{ for } Bn < Bn_{max}, \text{ and } \overline{Nu} = 1 \text{ for } Bn \ge Bn_{max} \quad (4)$$

where $A = aC_{\varphi 1}Ra^{m-0.25}[Pr^{n-0.25}/(1+Pr)^n] - 1/(Ra^{0.25}Pr^{0.25})$, $b = 0.025Ra^{0.25}Pr^{0.25}$ with $a = 0.178$, $m = 0.269$, $n = 0.02$ and $C_{\varphi 1} = 0.5^{-\varphi[rad]}$ which causes Eq. 4 to revert back to the expression for square enclosures as φ tends to zero. The expression given in Eq. 4 is dependent upon the adequate representation of Bn_{max}. An expression for Bn_{max} which extends upon a previous

expression proposed for square enclosures [2,6] to application in trapezoidal enclosures such that $Bn_{max} = (1 + C_{\varphi 2})[0.0019ln(Ra) - 0.0128]Ra^{0.55}Pr^{-0.50}$ where $C_{\varphi 2} = 4\varphi[rad]/\pi[rad]$. This expression for Bn_{max} tends back to the expression for square enclosures [2,6] as φ tends to zero. It is evident from Figs. 7a-c that the expression given by Eq. 4 generally provides satisfactory qualitative and mostly quantitative variation of $\overline{Nu}$ for the range of Ra, Bn and φ considered.

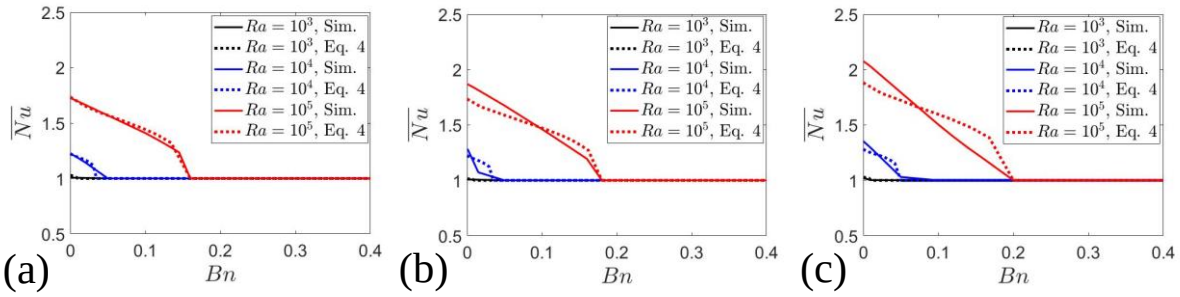

Figure 7: The variation of $\overline{Nu}$ normalised by value at Bn_{max} with Bn for $Ra = 10^3$, 10^4 and 10^5 where $Pr = 10^3$ for (a) $\varphi = 30°$, (b) $\varphi = 45°$, and (c) $\varphi = 60°$ along with the values from Eq. 4.

4. CONCLUSIONS

Laminar, steady-state, natural convection of Bingham fluids in trapezoidal enclosures with a heated bottom wall, adiabatic top and cooled inclined sidewalls has been analysed based on numerical simulations for a range of different values of Ra (i.e., $10^3 \leq Ra \leq 10^5$), Bn (i.e., $0.0 \leq Bn \leq 0.5$), and φ (i.e., $30° \leq \varphi \leq 60°$. It has been found that $\overline{Nu}$ increases with increasing Ra due to the strengthening of advective transport. However, an increase in the sidewall inclination φ leads to a decrease in $\overline{Nu}$. The value of $\overline{Nu}$ was found to decrease with increasing Bn value. At high values of Bn, the fluid flow becomes negligible within the enclosure and heat transfer begins to take place due to thermal conduction and, therefore, the value of $\overline{Nu}$ settles to a constant value irrespective of the value of Ra. Furthermore, a correlation for $\overline{Nu}$ for the considered configuration accounting for the range of Ra, Bn and φ has been proposed which provides satisfactory predictions of the qualitative variation of $\overline{Nu}$.

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